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Kuramoto-Sivashinsky Equation

Energy Bounds Through Scaling Analysis

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1 Introduction

In this thesis we are studying solutions $u(x,t)$ of the one-dimensional scalar Kuramoto-Sivashinsky equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \tag{KS}$$

which arises from a number of physical phenomena. It models two components reaction diffusion systems as studied by Kuramoto and Tsuzuki [KT76], the propagation of flame fronts as investigated by Sivashinsky [Siv80], the flow of a viscous liquid film on a vertical wall as analysed by Sivashinsky and Michelson [SM80] and crystal growth as discussed by Menke [Men00].

Behaviour of Solutions

Disregarding the non-linear term (KS) turns into the linearized Kuramoto-Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} = 0,$$

which under Fourier transformation becomes

$$\hat{u}_t = (\omega^2 - \omega^4)\hat{u},$$

implying that small frequencies $\omega < 1$ grow while high frequencies $\omega > 1$ get damped. The non-linear term however shifts energy from small to high frequencies, as can be seen in the study of Burgers' equation

$$u_t + uu_x = 0, \tag{BE}$$

which produces discontinuities from smooth initial data. This is demonstrated in Figure 1.1, where the evolution of sinusoidal initial data is plotted.

Together these two conflicting properties result in spatio-temporal chaotic behaviour of solutions of (KS), that can be seen in Figure 1.2.

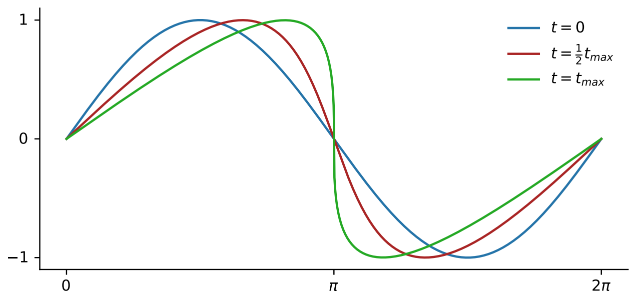

Figure 1.1: Sinusoidal initial data leads to discontinuities for solutions of Burgers' equation.

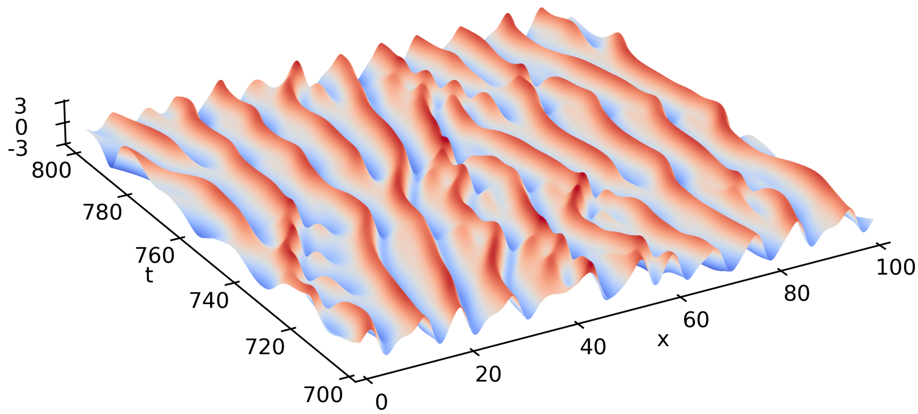

Figure 1.2: Typical chaotic behaviour of solutions of the Kuramoto-Sivashinsky equation.

Bounds for Solutions

In all the previously mentioned applications it is justified to assume spatially periodic solutions. The dynamics are particular interesting for large domains, so it is natural to investigate how solutions scale with respect to growing domains. For instance, in case of L-periodic solutions u, one is interested in bounds for the energy density

$$\frac{1}{L}\int_0^L u^2(x,t)\,dx$$

and the space-time energy density

$$\frac{1}{T}\int_0^T \frac{1}{L}\int_0^L u^2(x,t)\,dx\,dt$$

as $L \longrightarrow \infty$.

Nicolaenko, Scheurer and Temam [NST85] proved

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^2} = O\Big(L^{\frac{5}{2}}\Big), \tag{1.1}$$

where $\|\cdot\|_{L^2}$ is the standard, non-averaged L^2 norm defined by $\|f\|_{L^2}^2 = \int_0^L f^2\,dx$, for odd initial data through the background flow method similar to the approach outlined in Section 1.1 and carried out in Section 2.1 of Chapter 2. Collet, Eckmann, Epstein and Stubbe [Col+93] showed that the oddness assumption can be dropped and improved the estimate to

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^2} = O\Big(L^{\frac{8}{5}}\Big). \tag{1.2}$$

Viewing (KS) as a perturbation of (BE) as outlined in Section 1.2 and carried out in Section 2.2 of Chapter 2 Giacomelli and Otto [GO05] sharpened the bound to

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^2} = o\Big(L^{\frac{3}{2}}\Big). \tag{1.3}$$

Bronski and Gambill [BG06] proved the slightly weaker estimate

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^2} = O\Big(L^{\frac{3}{2}}\Big). \tag{1.4}$$

They also used the background flow method, which we will describe in Section 1.1 and prove in Section 2.1 of Chapter 2, and showed that (1.4) is the optimal bound for this approach as outlined in Section 1.1. This technique also yields applications to a wider range of equations such as the destabilized Kuramoto-Sivashinsky equation

$$u_t + \frac{1}{2}\left(u^2\right)_x + u_{xx} + u_{xxxx} = \gamma u,$$

where $\gamma > 0$ and may be applicable to the two-dimensional Kuramoto-Sivashinsky equation

$$u_t + \nabla u^2 + \Delta u + \Delta^2 u = 0, \qquad \nabla \times u = 0,$$

see [BG06, p.2037 f.]. Otto [Ott09] improved the previous bounds (1.1) - (1.4) to

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{L}\int_0^L \left(|\partial_x|^\alpha u(x,t)\right)^2 \, dx\, dt = \mathcal{O}\left(\ln^{\frac{5}{3}+} L\right) \tag{1.5}$$

for all $\frac{1}{3} < \alpha \leq 2$, where $|\partial_x|^\alpha u$ is the α-fractional derivative (see Definition A.1 in the Appendix) of u and the notation $\mathcal{O}\left(\ln^{\frac{5}{3}+} L\right)$ stats that the bound $\mathcal{O}(\ln^c L)$ holds for every $c > \frac{5}{3}$, which in this context yields

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \|u(\cdot,t)\|_{L^2} \, dt = \mathcal{O}(L^{\frac{5}{6}+}).$$

The most recent result,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{L}\int_0^L \left(|\partial_x|^\alpha u(x,t)\right)^2 \, dx\, dt = \mathcal{O}\left(\ln^{\frac{5}{6}+} L\right)$$

for all $\frac{1}{3} < \alpha \leq 2$, established by Goldman, Josien and Otto [GJO15] improves the initial result (1.5) of [Ott09] but does not yield a better bound for the time average of $\|u\|_{L^2}$.

However numerical simulations, such as by Wittenberg and Holmes [WH99] and Goluskin and Fantuzzi [GF19], suggest that the behaviour of solutions for domains that exceed a minimum length is independent of the domain size. This means that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty} = \mathcal{O}(1),$$

which would translate to

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^2} = \mathcal{O}(L^{\frac{1}{2}})$$

and

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \|u(\cdot,t)\|_{L^2}\, dt = \mathcal{O}(L^{\frac{1}{2}}).$$

With respect to the space and space-time average this would yield

$$\limsup_{t\to\infty} \frac{1}{L}\int_0^L (|\partial_x|^\alpha u(x,t))^2\, dx = \mathcal{O}(1)$$

and

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{L}\int_0^L (|\partial_x|^\alpha u(x,t))^2\, dx\, dt = \mathcal{O}(1)$$

for all $\alpha \geq 0$ as conjectured by Otto [Ott09].

Notation.
We write

- $f \lesssim g$ *if there exists a universal constant* $c > 0$ *such that* $f \leq cg$.
- $\int f\, dx$ *if the domain of integration matches with the periodicity of the domain, i.e. for* L*-periodic* f

$$\int f(x)\, dx = \int_s^{s+L} f(x)\, dx$$

for any $s \in \mathbb{R}$. *Similarly for Lebesgue, Hilbert and Sobolev norms*

$$\|f\|_{L^p} = \|f\|_{L^p(s,s+L)}, \quad \|f\|_{H^k} = \|f\|_{H^k(s,s+L)}, \quad \|f\|_{W^{k,p}} = \|f\|_{W^{k,p}(s,s+L)}.$$

- *c for a generic constant that may differ in consecutive estimates.*
- *$\mathcal{D}(\Omega)$ for the space $C_c^\infty(\Omega)$ of smooth functions with compact support in Ω.*
- *$A^\star$ for the dual space of A.*

1.1 The Background Flow Method

In Section 2.1 of Chapter 2 we will follow Bronski and Gambill [BG06] in deriving bounds for solutions $u(x,t)$ of the initial boundary value problem

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \tag{KS}$$
$$u(x-L,t) = u(x+L,t), \tag{PC$_{2L}$}$$
$$u(0,t) = 0, \tag{BC$_0$}$$
$$u(x,0) = u_0(x) \tag{IC}$$

for all $x \in \mathbb{R}$ and $t > 0$. The initial condition is chosen to match the periodicity, the boundary condition, to be locally square-integrable and have zero mean, i.e.

$$\begin{aligned} u_0(x-L) &= u_0(x+L), \\ u_0(0) &= 0, \\ \int_{-L}^{L} u_0(x)\,dx &= 0, \\ u_0 &\in L^2(-L,L) \end{aligned}$$

for all $x \in \mathbb{R}$.

For this we will construct a $2L$ periodic function φ_x with zero mean (see Theorem 2.7 for the construction and Figure 2.2 for a plot of φ_x) and $\|\varphi\|_{H^2} \leq cL^{\frac{3}{2}}$ (see Corollary 2.8) such that

$$\int u_{xx}^2 - u_x^2 + \varphi_x u^2\,dx \geq \frac{1}{4}\int u_{xx}^2 + u^2\,dx \tag{1.6}$$

holds for every $2L$ periodic $u \in C^3[-L,L]$ with $u(0) = 0$, which implies for the operator $K = \partial_{xxxx} + \partial_{xx} + \varphi_x$ that the form

$$\langle u, Ku\rangle = \int u_{xx}^2 - u_x^2 + \varphi_x u^2 \tag{1.7}$$

is coercive, i.e. there exists a constant $\lambda_0 > 0$ such that

$$\langle u, Ku\rangle \geq \lambda_0 \|u\|_{L^2}^2. \tag{1.8}$$

Linearly rescaling solutions $\tilde{u}(x,t) = u(2x,t) = u(\tilde{x},t)$ of (KS), (PC_{2L}) and the potential according to $\tilde{\varphi}(x) = \frac{1}{8}\varphi(\tilde{x})$, Lemma 2.2 and, after establishing the regularity requirements, the coercivity result yield the Lyapunov function

$$\begin{aligned} c\frac{d}{dt}\int (\tilde{u} - 8\tilde{\varphi})^2 \, d\tilde{x} &\leq -\langle \tilde{u}, K\tilde{u}\rangle + \int \tilde{\varphi}_{\tilde{x}}^2 + \tilde{\varphi}_{\tilde{x}\tilde{x}}^2 \, d\tilde{x} \\ &\leq -\lambda_0 \|\tilde{u}\|_{L^2}^2 + \int \tilde{\varphi}_{\tilde{x}}^2 + \tilde{\varphi}_{\tilde{x}\tilde{x}}^2 \, d\tilde{x}. \end{aligned} \tag{1.9}$$

Grönwall Inequality (see Proposition A.10 in the Appendix) implies that the ball $B_r(0) \subset L^2$ is an exponentially attracting set for $\tilde{u}$, as

$$\operatorname{dist}(\tilde{u}(\cdot,t), B_r(0)) \leq e^{-\frac{\lambda_0}{c}t}\|\tilde{u}_0\|_{L^2},$$

where its radius can be bounded by $r \leq c\|\tilde{\varphi}\|_{H^2}$. Finally in Corollary 2.9 this results in the estimate

$$\limsup_{t\to\infty} \|u\|_{L^2} = \mathcal{O}(L^{\frac{3}{2}}).$$

Remark on the Optimality of Bounds for this Method

In [NST85], [Col+93] and [Goo94] the constructed $2L$-periodic function φ_x is given by

$$\varphi_x(x) = \gamma L^{c_2-c_1-1} + q(x) = \gamma L^{c_2-c_1-1} + L^{c_2}\tilde{q}(L^{c_1}x)$$

with constants $\gamma, c_1, c_2 > 0$, where $\tilde{q}(x) \in C^2$ is compactly supported on an interval independent of L and therefore $q(x) \in C_c^2(-L,L)$ for $c_1 \leq 1$ and L sufficiently large. In [NST85, Equation (2.13b)] φ_x is defined via a Fourier series such that $c_2 = 2$ and $c_1 = 1$, while in [Col+93, Equation (3.1)] it is chosen similarly implying $c_2 = \frac{7}{5}$ and $c_1 = \frac{2}{5}$. Defining $q(x)$ as a mollification of

$$\begin{cases} cL^2(Lx)^2 & \text{for } |x| \leq \epsilon \\ 0 & \text{for } |x| > \epsilon \end{cases}$$

implies that q fulfills the requirements [Goo94, Equations (2.6)-(2.10)] with $\epsilon = \tilde{c}L^{-1}$, such that the exponents for the function defined in [Goo94, Equation (2.3)] are also given by $c_2 = 2$ and $c_1 = 1$.

Setting $u(x) = L^{-\frac{1}{2}} \sin\left(\frac{k\pi x}{L}\right)$, where $k = \left\lfloor \frac{L}{2\pi} \right\rfloor$, the form becomes

$$\begin{aligned}\langle u, Ku\rangle &= \frac{1}{L}\int \sin^2\left(\frac{k\pi x}{L}\right)\left(L^{-4}(k\pi)^4 - L^{-2}(k\pi)^2 + \gamma L^{c_2-c_1-1} + q(x)\right) dx \\ &\leq \frac{1}{L}\int \sin^2\left(\frac{k\pi x}{L}\right)\left(-\frac{3}{16} + \gamma L^{c_2-c_1-1} + q(x)\right) dx.\end{aligned}$$

If $c_2 - c_1 - 1 < 0$ then $\gamma L^{c_2-c_1-1} \longrightarrow 0$ as $L \longrightarrow \infty$ and since φ_x has zero average also $q(x) \longrightarrow 0$ implying that there exists some L such that $\langle u, Ku\rangle < 0$.

Rescaling u according to $\tilde{u}(y,t) = \tilde{u}(L^{c_1}x, t) = u(x,t)$ we get

$$\begin{aligned}\langle u, Ku\rangle &= \int_{-L}^{L} u_{xx}^2 - u_x^2 + \varphi_x u^2 \\ &= \frac{1}{L^{c_1}}\int_{-L^{1+c_1}}^{L^{1+c_1}} L^{4c_1}u_{yy}^2\left(\frac{y}{L^{c_1}}\right) - L^{2c_1}u_y^2\left(\frac{y}{L^{c_1}}\right) + q\left(\frac{y}{L^{c_1}}\right)u^2\left(\frac{y}{L^{c_1}}\right) \\ &\qquad + \gamma L^{c_2-c_1-1}u\left(\frac{y}{L^{c_1}}\right)dy \\ &= L^{3c_1}\int_{-L^{1+c_1}}^{L^{1+c_1}} \tilde{u}_{yy}^2(y) - L^{-2c_1}\tilde{u}_{yy}^2 + L^{c_2-4c_1}\tilde{q}\tilde{u}^2 + \gamma L^{c_2-5c_1-1}\tilde{u}^2\, dy.\end{aligned}$$

Regarding the term $L^{c_2-4c_1}\tilde{q}\tilde{u}^2$, the potential is called strong if $c_2 - 4c_1 > 0$, weak if $c_2 - 4c_1 < 0$ and critical if $c_2 = 4c_1$. The functions constructed in [NST85], [Col+93] and [Goo94] are all weak potentials. Since $\varphi_x \in C^2$ has zero mean and $\gamma > 0$ there exists a neighbourhood where $\tilde{q} < 0$. Let $\tilde{u}$ be compactly supported in this neighbourhood, then a strong potential would imply

$$\begin{aligned}\langle u, Ku\rangle &= L^{3c_1}\int_{-L^{1+c_1}}^{L^{1+c_1}} \tilde{u}_{yy}^2(y) - L^{-2c_1}\tilde{u}_{yy}^2 + L^{c_2-4c_1}\tilde{q}\tilde{u}^2 + \gamma L^{c_2-5c_1-1}\tilde{u}^2\, dy \\ &\leq -cL^{c_2-c_1} + O(L^{3c_1}) + O(L^{c_1}) + O(L^{c_2-2c_1-1}) \\ &< 0\end{aligned}$$

for some $L > 0$ and therefore contradict the coercivity requirement.

Calculating

$$\begin{aligned}\|\varphi\|_{H^2}^2 &\geq \|\varphi_{xx}\|_{L^2}^2 \\ &= \int_{-L}^{L} (q'(x))^2 dx \\ &= L^{2c_2} \int_{L^{c_1}x \in \operatorname{supp}\tilde{q}} \left(\frac{d}{dx}\tilde{q}(L^{c_1}x)\right)^2 dx \\ &= L^{2c_2+2c_1} \int_{L^{c_1}x \in \operatorname{supp}\tilde{q}} (\tilde{q}'(L^{c_1}x))^2 dx \\ &= L^{2c_2+c_1} \int_{u \in \operatorname{supp}\tilde{q}} (\tilde{q}'(u))^2 du \\ &= L^{2c_2+c_1} c\end{aligned}$$

implies the lower bound $\|\varphi\|_{H^2} \gtrsim L^{c_2+\frac{c_1}{2}}$ such that the optimal potential minimizes $c_2 + \frac{c_1}{2}$ under the constraints $c_2 \leq 4c_1$ and $c_2 \geq c_1 + 1$. This minimum is achieved by $c_1 = \frac{1}{3}$ and $c_2 = \frac{4}{3}$, which coincides with the potential we will construct implying that it yields the optimal bound for this method.

Remark on the Initial Conditions

In most applications, such as the flame propagation in [Siv80], the chemical reaction diffusion behaviour in [KT76] and the crystal growth in [Men00], the Kuramoto-Sivashinsky equation is derived from a quantity h satisfying an equation of type

$$h_t + \frac{1}{2}h_x^2 + h_{xx} + h_{xxxx} = 0.$$

From this (KS) can be obtained by taking the derivative in space and setting $u = h_x$. If h_0 is also periodic, then u has zero mean for all times since

$$\frac{d}{dt}\int_{-L}^{L} u(x,t)\,dx = \int \frac{1}{2}(u^2)_x + u_{xx} + u_{xxxx}\,dx = 0$$

and therefore

$$\int_{-L}^{L} u(x,t)\,dx = \int_{-L}^{L} u_0\,dx = \int_{-L}^{L} (h_0)_x\,dx = 0.$$

In [NST85] and [Col+93, Chapter 2] the constructed potential requires odd solutions. This requirement was removed in [Col+93, Chapters 4 and 5], where an arbitrary periodic zero average function u was split into

$$u(x) = u(0) + u_{\text{even}}(x) + u_{\text{odd}}(x)$$

with even u_{even} such that $u_{\text{even}}(0) = 0$ and odd u_{odd}. For $u_{\text{even}}(0)$ an estimate similar to (1.6) was found and the potential was shifted by a function of time as $\varphi(x + b(t))$ resulting in an estimate similar to (1.9). The bounds presented here only require $u(0, t) = 0$ such that the outlined approach works for odd solutions and can therefore also be generalized to $2L$-periodic, locally square-integrable initial data with zero mean.

1.2 Considering Kuramoto-Sivashinsky as a Perturbation of Burgers' Equation

Based on the work of Giacomelli and Otto [GO05] we will derive bounds for the solution $u(x, t)$ of the initial boundary value problem

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \tag{KS}$$

$$u(x, t) = u(x + L, t), \tag{PC$_\text{L}$}$$

$$u(x, 0) = u_0(x) \tag{IC}$$

for all $x \in \mathbb{R}$ and $t > 0$. The initial condition is chosen to match the periodicity condition, to be locally square-integrable and have zero mean, i.e.

$$\begin{aligned} u_0(x) &= u_0(x + L), \\ \int_0^L u_0(x)\, dx &= 0, \\ u_0 &\in L^2(0, L) \end{aligned} \tag{1.10}$$

for all $x \in \mathbb{R}$.

Under the rescaling $L\hat{u}(\hat{x}, \hat{t}) = u(L\hat{x}, \hat{t}) = u(x, t)$ the Kuramoto-Sivashinsky equation (KS) itself becomes

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} + \frac{1}{L^2}\hat{u}_{\hat{x}\hat{x}} + \frac{1}{L^4}\hat{u}_{\hat{x}\hat{x}\hat{x}\hat{x}} = 0$$

and using (KS) one gets

$$\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x - 2(u_x^2)_{xx} + \left(\frac{1}{2}u^2\right)_{xxxx} = \frac{1}{4}u^2 - \left(\frac{1}{2}u + u_{xx}\right)^2 \le \frac{1}{4}u^2,$$

for which the rescaling implies

$$\left(\frac{1}{2}\hat{u}^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}^3\right)_{\hat{x}} - \frac{2}{L^4}(\hat{u}_{\hat{x}}^2)_{\hat{x}\hat{x}} + \frac{1}{L^4}\left(\frac{1}{2}\hat{u}^2\right)_{\hat{x}\hat{x}\hat{x}\hat{x}} \le \frac{1}{4}\hat{u}^2.$$

For $L \longrightarrow \infty$ this yields (see Theorem 2.16) that the rescaled solution of (KS) distributionally solves the Burgers' equation

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 0 \tag{BE}$$

and the entropy condition

$$\left(\frac{1}{2}\hat{u}^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}^3\right)_{\hat{x}} \le \frac{1}{4}\hat{u}^2. \tag{EC$_{\hat{u}^2}$}$$

Conservation laws

$$u_t + f(u)_x = 0 \tag{1.11}$$

such as (KS) with $f(u) = \frac{1}{2}u^2 + u_x + u_{xxx}$ or (BE) with $f(u) = \frac{1}{2}u^2$ are strongly related to their corresponding Hamilton-Jacobi equations

$$h_t + f(h_x) = 0. \tag{1.12}$$

In order to see how one derives (1.12) from (1.11) we write the conservation law via the space-time curl $\begin{pmatrix} -\partial_x \\ \partial_t \end{pmatrix}$ as

$$\begin{aligned} 0 &= u_t + f(u)_x \\ &= \begin{pmatrix} -\partial_x \\ \partial_t \end{pmatrix} \cdot \begin{pmatrix} -f(u) \\ u \end{pmatrix} \\ &= \operatorname{curl}\begin{pmatrix} -f(u) \\ u \end{pmatrix}. \end{aligned}$$

So the vector field $\begin{pmatrix} u \\ -f(u) \end{pmatrix}$ is curl-free and therefore conservative. Using Helmholtz decomposition there exists a potential h such that

$$\begin{pmatrix} h_x \\ h_t \end{pmatrix} = \nabla h = \begin{pmatrix} u \\ -f(u) \end{pmatrix},$$

which implies that h solves the Hamilton-Jacobi equation

$$h_t = -f(u) = -f(h_x).$$

Deriving the Hamilton-Jacobi equation with respect to space and setting $u = h_x$ one regains the corresponding conservation law (1.11).

Using this relation De Lellis, Otto and Westdickenberg showed in [DOW04, Thoerem 2.4] that if $\hat{u}$ solves (BE) and (EC$_{\hat{u}^2}$), where the right-hand side could even be replaced by a measure $\hat{\mu}$ with

$$\lim_{r \to 0} \frac{1}{r} \hat{\mu}(B_r(\hat{x}, \hat{t})) = 0 \tag{1.13}$$

for all x, t, then the associated function $\hat{h}$ is a viscosity solution (see Definition 1.2) of the corresponding Hamilton-Jacobi equation

$$\hat{h}_{\hat{t}} + \frac{1}{2} \hat{h}_{\hat{x}}^2 = 0.$$

This is carried out in Theorem 2.19, where we show that $\hat{h}$ is a viscosity subsolution by mollification and the rather technical argument to prove that $\hat{h}$ is also a viscosity supersolution is based on estimates for the average of functions around the required minimum condition.

In return [DOW04, Corollary 2.5] yields that $\hat{u}$ is an entropy solution (see Definition 1.1) of $\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 0$ so that $\hat{u}$ solves

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 0, \tag{BE}$$

$$\left(\frac{1}{2}\hat{u}^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}^3\right)_{\hat{x}} \leq 0, \tag{EC$_0$}$$

implying that the right-hand side of (EC$_{\hat{u}^2}$) is negligible. The argument here is that since $\hat{h}$ is a viscosity solution, $\hat{u}$ can only have decreasing jumps, which implies (EC_0) as shown in Corollary 2.20.

For smooth, periodic solutions $\hat{u}$ of (BE) the energy would be preserved since

$$\frac{d}{d\hat{t}}\int \hat{u}^2\, d\hat{x} = \int \hat{u}\hat{u}_{\hat{t}}\, d\hat{x} \overset{\text{(BE)}}{=} -\int \hat{u}^2\hat{u}_{\hat{x}}\, d\hat{x} = -\frac{1}{3}\int \left(\hat{u}^3\right)_{\hat{x}}\, d\hat{x} = 0,$$

but in our case $\hat{u}$ rather has L^4_{loc}-regularity and fulfills (BE) and (EC_0) in $\mathcal{D}^\star$, which implies that energy is dissipated and one finds the bound

$$\int \hat{u}^2(\hat{x},\hat{t})\, d\hat{x} \leq \frac{c}{\hat{t}^2} \tag{1.14}$$

as carried out in Lemma 2.21. Translating this estimate back to the original solution we get

$$\int u^2(x,t)\, dx \leq cL^3\left(1+\frac{1}{t^2}\right)$$

via contradiction in Corollary 2.22 resulting in (see Corollary 2.25)

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^2} = o\left(L^{\frac{3}{2}}\right), \tag{1.15}$$

the bound claimed in (1.3).

▶ **Definition 1.1 (Convex Entropy-Entropy Flux Pair, Entropy Solution).** For $\Omega \subset \mathbb{R}^2, f \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$

- $\eta, q \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ is a convex entropy-entropy flux pair if
  1. η is convex,
  2. $q' = f'\eta'$ almost everywhere.
- $\hat{u} \in L^1_{\text{loc}}(\Omega)$ is an entropy solution of $\hat{u}_{\hat{t}} + f(\hat{u})_{\hat{x}} = 0$ if
  1. $\hat{u}_{\hat{t}} + f(\hat{u})_{\hat{x}} = 0$,
  2. $\eta(\hat{u})_{\hat{t}} + q(\hat{u})_{\hat{x}} \leq 0$

  hold in $\mathcal{D}^\star$ for every convex entropy-entropy flux pair. ◀

Remark on the Definition of Entropy Solutions

Based on [Eva98, p.604] suppose that $\hat{u}$ is smooth, then $\hat{u}_{\hat{t}} + f(\hat{u})_{\hat{x}} = 0$ and

$$\begin{aligned}\eta(\hat{u})_{\hat{t}} + q(\hat{u})_{\hat{x}} &= \eta'(\hat{u})\hat{u}_{\hat{t}} + q'(\hat{u})\hat{u}_{\hat{x}} \\ &= \eta'(\hat{u})(\hat{u}_{\hat{t}} + f'(\hat{u})\hat{u}_{\hat{x}}) \\ &= \eta'(\hat{u})(\hat{u}_{\hat{t}} + f(\hat{u})_{\hat{x}}) \\ &= 0.\end{aligned}$$

In physical applications $\eta(\hat{u})$ often donates the negative entropy, while $q(\hat{u})$ is the entropy flux, giving them the name entropy-entropy flux pair. Let $\Omega^{\hat{x}}$ be a spatial domain in Ω, then the entropy $-\eta$ in this domain evolves as

$$\begin{aligned}\frac{d}{d\hat{t}} \int_{\Omega^{\hat{x}}} -\eta \; d\hat{x} &= \int_{\Omega^{\hat{x}}} -\eta_{\hat{t}} \; d\hat{x} \\ &= \int_{\Omega^{\hat{x}}} q_{\hat{x}} \; d\hat{x} \\ &= q\Big|_{\partial\Omega^{\hat{x}}},\end{aligned}$$

implying the entropy of the domain changes according to the flow on the boundary. Since for non-smooth solutions the inequality yields that the change of entropy in the domain can not be smaller than the flow on the boundary this states that the entropy of a closed system can only increase and therefore agrees with the second law of thermodynamics.

▶ **Definition 1.2 (Viscosity Solution).**

Let $\Omega \subset \mathbb{R}^2$. $\hat{h} \in C(\Omega)$ is called

- a viscosity subsolution of $\hat{h}_{\hat{t}} + f(\hat{h}_{\hat{x}}) = 0$ if for all $\xi \in C^\infty(\Omega)$ the following holds:

  – If $\hat{h} - \xi$ has a local maximum at a point $(\hat{x}, \hat{t}) \in \Omega$, then

  $$\xi_{\hat{t}}(\hat{x}, \hat{t}) + f(\xi_{\hat{x}}(\hat{x}, \hat{t})) \leq 0.$$

- a viscosity supersolution of $\hat{h}_{\hat{t}} + f(\hat{h}_{\hat{x}}) = 0$ if for all $\xi \in C^\infty(\Omega)$ the following holds:

  – If $\hat{h} - \xi$ has a local minimum at a point $(\hat{x}, \hat{t}) \in \Omega$, then

$$\xi_{\hat{t}}(\hat{x}, \hat{t}) + f(\xi_{\hat{x}}(\hat{x}, \hat{t})) \geq 0.$$

- a viscosity solution of $\hat{h}_{\hat{t}} + f(\hat{h}_{\hat{x}}) = 0$ if it is a viscosity sub- and supersolution of $\hat{h}_{\hat{t}} + f(\hat{h}_{\hat{x}}) = 0$. ◀

Remark on the Definition of Viscosity Solutions
The name viscosity solution seems misleading since there is no viscosity term in the definition, but it arises from the method of vanishing viscosity as stated in [Eva98, ch. 10.1].

Instead of looking at solutions $\hat{h}$ of

$$\hat{h}_{\hat{t}} + f(\hat{h}_{\hat{x}}) = 0$$

one considers $\epsilon > 0$ and looks at solutions $\hat{h}^\epsilon$ of

$$\hat{h}^\epsilon_{\hat{t}} + f(\hat{h}^\epsilon_{\hat{x}}) - \epsilon \hat{h}^\epsilon_{\hat{x}\hat{x}} = 0, \tag{1.16}$$

which are smooth since the viscosity term $h^\epsilon_{\hat{x}\hat{x}}$ has a regularizing effect as can be seen by Fourier analysis. Because of compactness results one gets that if $\hat{h} - \xi$ has a maximum in $(\hat{x}, \hat{t})$ then so does $\hat{h}^\epsilon - \xi$ in $(\hat{x}_\epsilon, \hat{t}_\epsilon) \longrightarrow (\hat{x}, \hat{t})$. This yields

$$\begin{aligned} \hat{h}^\epsilon_{\hat{x}}(\hat{x}_\epsilon, \hat{t}_\epsilon) &= \xi_{\hat{x}}(\hat{x}_\epsilon, \hat{t}_\epsilon), \\ \hat{h}^\epsilon_{\hat{t}}(\hat{x}_\epsilon, \hat{t}_\epsilon) &= \xi_{\hat{t}}(\hat{x}_\epsilon, \hat{t}_\epsilon) \end{aligned} \tag{1.17}$$

and

$$-\hat{h}^\epsilon_{\hat{x}\hat{x}}(\hat{x}_\epsilon, \hat{t}_\epsilon) \geq -\xi_{\hat{x}\hat{x}}(\hat{x}_\epsilon, \hat{t}_\epsilon) \tag{1.18}$$

such that

$$\begin{aligned} \xi_{\hat{t}}(\hat{x}_\epsilon, \hat{t}_\epsilon) + f(\xi_{\hat{x}}(\hat{x}_\epsilon, \hat{t}_\epsilon)) &\overset{(1.17)}{=} \hat{h}^\epsilon_{\hat{t}}(\hat{x}_\epsilon, \hat{t}_\epsilon) + f(\hat{h}^\epsilon_{\hat{x}}(\hat{x}_\epsilon, \hat{t}_\epsilon)) \\ &\overset{(1.16)}{=} \epsilon \hat{h}^\epsilon_{\hat{x}\hat{x}}(\hat{x}_\epsilon, \hat{t}_\epsilon) \\ &\overset{(1.18)}{\leq} \epsilon \xi_{\hat{x}\hat{x}}(\hat{x}_\epsilon, \hat{t}_\epsilon). \end{aligned}$$

Now letting $\epsilon \to 0$ we get, with $(\hat{x}_\epsilon, \hat{t}_\epsilon) \to (x,t)$ and since ξ is smooth, the definition of a viscosity subsolution. Similarly one gets the supersolution through reversed inequality signs.

Remark on the Different Periodicity Lengths

As stated earlier we are interested in the scaling of solution with respect to the order of the underlying domain size and not what the constants for the bounds are. Therefore the difference coming from linear scaling of the periodicity condition such as $2L$ in (PC_{2L}) instead of L in (PC_L) or the $4L$-periodicity of $\tilde{u}$ in Lemma 2.2 is irrelevant for these bounds.

2 Derivation of the Bounds

2.1 Using Background Flow Method

Based on [BG06] in this Section we carry out the approach outlined in Section 1.1 of Chapter 1. In Section 2.1.1 we justify why the solution can be bounded by the potential if the coercivity result holds. In Section 2.1.2 we construct the potential and in Section 2.1.3 we get the actual bound.

2.1.1 Conditions for an Attracting Region

First we find a condition for which $B_r(0)$, the ball of radius r around the origin in L^2, exponentially attracts functions $\tilde{u} \in L^2$.

▶ **Lemma 2.1.**
Let $\tilde{u} = \tilde{u}(\cdot, t) \in L^2[-\tilde{L}, \tilde{L}]$ and $\tilde{\varphi} = \tilde{\varphi}(\tilde{x}) \in L^2[-\tilde{L}, \tilde{L}]$ satisfy

$$\frac{d}{dt}\|\tilde{u} - \tilde{\varphi}\|_{L^2}^2 \leq -\lambda_0\|\tilde{u}\|_{L^2}^2 + P^2 \tag{2.1}$$

for some constant $\lambda_0 > 0$ and P independent of t, then

$$\operatorname{dist}(\tilde{u}(\cdot, t), B_r(0)) \leq e^{-\frac{\lambda_0}{4}t}\|\tilde{u}_0\|_{L^2},$$

where $r = \sqrt{2\|\tilde{\varphi}\|_{L^2}^2 + \frac{2P^2}{\lambda_0}} + 2\|\tilde{\varphi}\|_{L^2}$ and $B_r(0)$ donates the L^2-ball around 0 with radius r. ◀

Proof.
The triangular inequality and Cauchy inequality (see Proposition A.3 in the Appendix) imply

$$\lambda_0\|\tilde{u} - \tilde{\varphi}\|_{L^2}^2 \leq \lambda_0(\|\tilde{u}\|_{L^2} + \|\tilde{\varphi}\|_{L^2})^2 \overset{(A.3)}{\leq} 2\lambda_0\|\tilde{u}\|_{L^2}^2 + 2\lambda_0\|\tilde{\varphi}\|_{L^2}^2 \tag{2.2}$$

such that

$$\frac{d}{dt}\|\tilde{u}-\tilde{\varphi}\|_{L^2}^2 \overset{(2.1)}{\leq} -\lambda_0\|\tilde{u}\|_{L^2}^2 + P^2 \overset{(2.2)}{\leq} -\frac{\lambda_0}{2}\|\tilde{u}-\tilde{\varphi}\|_{L^2}^2 + \lambda_0\|\tilde{\varphi}\|_{L^2}^2 + P^2.$$

Grönwall inequality (see Proposition A.10 in the Appendix) yields

$$\begin{aligned}
\|\tilde{u}-\tilde{\varphi}\|_{L^2}^2 &\overset{(A.10)}{\leq} e^{-\frac{\lambda_0}{2}t}\left(\|\tilde{u}_0-\tilde{\varphi}\|_{L^2}^2 + \int_0^t e^{\int_0^s \frac{\lambda_0}{2}\,dr}\left(\lambda_0\|\tilde{\varphi}\|_{L^2}^2 + P^2\right)ds\right) \\
&= e^{-\frac{\lambda_0}{2}t}\left(\|\tilde{u}_0-\tilde{\varphi}\|_{L^2}^2 + \frac{2}{\lambda_0}\left(\lambda_0\|\tilde{\varphi}\|_{L^2}^2 + P^2\right)\left(e^{\frac{\lambda_0}{2}t}-1\right)\right) \\
&\leq e^{-\frac{\lambda_0}{2}t}\|\tilde{u}_0-\tilde{\varphi}\|_{L^2}^2 + 2\|\tilde{\varphi}\|_{L^2}^2 + \frac{2P^2}{\lambda_0}.
\end{aligned}$$

Using $a^2+b^2 \leq a^2+2ab+b^2 = (a+b)^2$ for $a,b \geq 0$ with $a = e^{-\frac{\lambda_0}{4}t}\|\tilde{u}_0-\tilde{\varphi}\|_{L^2}$ and $b = \sqrt{2\|\tilde{\varphi}\|_{L^2}^2 + \frac{2P^2}{\lambda_0}}$ we get

$$\begin{aligned}
\|\tilde{u}-\tilde{\varphi}\|_{L^2} &\leq e^{-\frac{\lambda_0}{4}t}\|\tilde{u}_0-\tilde{\varphi}\|_{L^2} + \sqrt{2\|\tilde{\varphi}\|_{L^2}^2 + \frac{2P^2}{\lambda_0}} \\
&\leq e^{-\frac{\lambda_0}{4}t}\|\tilde{u}_0\|_{L^2} + \|\tilde{\varphi}\|_{L^2} + \sqrt{2\|\tilde{\varphi}\|_{L^2}^2 + \frac{2P^2}{\lambda_0}}
\end{aligned}$$

such that for $r_1 = \|\tilde{\varphi}\|_{L^2} + \sqrt{2\|\tilde{\varphi}\|_{L^2}^2 + \frac{2P^2}{\lambda_0}}$

$$\begin{aligned}
\|\tilde{u}-\tilde{\varphi}\|_{L^2} &\leq e^{-\frac{\lambda_0}{4}t}\|\tilde{u}_0\|_{L^2} + r_1, \\
\operatorname{dist}(\tilde{u}, B_{r_1}(\tilde{\varphi})) &\leq e^{-\frac{\lambda_0}{4}t}\|\tilde{u}_0\|_{L^2}
\end{aligned} \tag{2.3}$$

and since by triangular inequality $B_{r_1}(\tilde{\varphi}) \subset B_r(0)$ for $r = r_1 + \|\tilde{\varphi}\|_{L^2} = 2\|\tilde{\varphi}\|_{L^2} + \sqrt{2\|\tilde{\varphi}\|_{L^2}^2 + \frac{2P^2}{\lambda_0}}$ we arrive at

$$\begin{aligned}
\operatorname{dist}(\tilde{u}, B_r(0)) &= \inf_{\xi\in B_r(0)} \operatorname{dist}(\tilde{u}, \xi) \\
&\leq \inf_{\xi\in B_{r_1}(\tilde{\varphi})} \operatorname{dist}(\tilde{u}, \xi)
\end{aligned}$$

$$= \operatorname{dist}(\tilde{u}, B_{r_1}(\tilde{\varphi}))$$
$$\overset{(2.3)}{\leq} e^{-\frac{\lambda_0}{4}t}\|\tilde{u}_0\|_{L^2},$$

concluding the proof. ■

Now we set this $\tilde{u}$ to be a rescaling of u in order to eliminate the constants and match the form described in Equation (1.7).

▶ **Lemma 2.2.**
Let $u(\cdot, t) \in H^2$ solve (KS), ($\text{PC}_{2\text{L}}$) and rescale it according to

$$\tilde{x} = 2x,$$
$$\tilde{u}(x,t) = u(2x,t),$$

then

$$\frac{1}{16}\frac{d}{dt}\int_{-2L}^{2L}(\tilde{u} - 8\tilde{\varphi})^2\,d\tilde{x} \leq \int_{-2L}^{2L}\tilde{u}_{\tilde{x}}^2 - \tilde{u}_{\tilde{x}\tilde{x}}^2 - \tilde{\varphi}_{\tilde{x}}\tilde{u}^2\,d\tilde{x} + \int_{-2L}^{2L} 8\tilde{\varphi}_{\tilde{x}}^2 + 64\tilde{\varphi}_{\tilde{x}\tilde{x}}^2\,d\tilde{x}$$

holds for all $\tilde{\varphi} \in H^2_{\text{per}}[-2L, 2L]$ with zero mean. ◀

Proof.
We set

$$\varphi(x) = 8\tilde{\varphi}\left(\frac{1}{2}x\right)$$

to match the spatial scaling of u. Because of the periodicity, integrating by parts yields

$$\int_{-L}^{L} u^2 u_x\,dx = \frac{1}{3}\int_{-L}^{L}(u^3)_x\,dx = 0,$$
$$\int_{-L}^{L}\varphi u u_x\,dx = \frac{1}{2}\int_{-L}^{L}\varphi(u^2)_x\,dx = -\frac{1}{2}\int_{-L}^{L}\varphi_x u^2\,dx \tag{2.4}$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\|u - \varphi\|^2_{L^2(-L,L)} = \int_{-L}^{L}(u - \varphi)u_t\,dx$$

$$
\begin{aligned}
&\overset{\text{(KS)}}{=} \int_{-L}^{L} (u-\varphi)(-uu_x - u_{xx} - u_{xxxx})\, dx \\
&= \int_{-L}^{L} -u^2 u_x - uu_{xx} - uu_{xxxx} + \varphi uu_x + \varphi u_{xx} + \varphi u_{xxxx}\, dx \\
&\overset{(2.4)}{=} \int_{-L}^{L} -uu_{xx} - uu_{xxxx} - \frac{1}{2}\varphi_x u^2 + \varphi u_{xx} + \varphi u_{xxxx}\, dx \\
&= \int_{-L}^{L} u_x^2 - u_{xx}^2 - \frac{1}{2}\varphi_x u^2 - \varphi_x u_x + \varphi_{xx} u_{xx}\, dx.
\end{aligned} \tag{2.5}
$$

Applying Cauchy inequality (see Proposition A.3 in the Appendix) with $a_1 = -\varphi_x$, $b_1 = u_x$ and $a_2 = \varphi_{xx}$, $b_2 = u_{xx}$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-\varphi\|^2_{L^2(-L,L)} \\
&\quad \overset{(2.5)}{=} \int_{-L}^{L} u_x^2 - u_{xx}^2 - \frac{1}{2}\varphi_x u^2 - \varphi_x u_x + \varphi_{xx} u_{xx}\, dx \\
&\quad \overset{(A.3)}{\leq} \int_{-L}^{L} u_x^2 - u_{xx}^2 - \frac{1}{2}\varphi_x u^2 + \frac{1}{2}c_1\varphi_x^2 + \frac{1}{2c_1}u_x^2 + \frac{1}{2}c_2\varphi_{xx}^2 + \frac{1}{2c_2}u_{xx}^2\, dx \\
&\quad = \int_{-L}^{L} \left(1 + \frac{1}{2c_1}\right)u_x^2 + \left(\frac{1}{2c_2} - 1\right)u_{xx}^2 - \frac{1}{2}\varphi_x u^2 + \frac{1}{2}c_1\varphi_x^2 + \frac{1}{2}c_2\varphi_{xx}^2\, dx.
\end{aligned} \tag{2.6}
$$

The substitution

$$
\begin{aligned}
\tilde{x} &= \beta x, \\
\tilde{u}(\beta x, t) &= u(x,t), \\
\tilde{\varphi}(\beta x) &= \gamma \varphi(x)
\end{aligned}
$$

implies

$$
\begin{aligned}
&\frac{1}{2\beta}\frac{d}{dt}\int_{-\beta L}^{\beta L} \left(\tilde{u}(\tilde{x}) - \frac{1}{\gamma}\tilde{\varphi}(\tilde{x})\right)^2 d\tilde{x} \\
&\quad = \frac{1}{2}\frac{d}{dt}\int_{-L}^{L} \left(\tilde{u}(\beta x) - \frac{1}{\gamma}\tilde{\varphi}(\beta x)\right)^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{d}{dt}\|u-\varphi\|^2_{L^2(-L,L)} \\
&\overset{(2.6)}{\leq} \int_{-L}^{L} \left(1+\frac{1}{2c_1}\right)u_x^2 + \left(\frac{1}{2c_2}-1\right)u_{xx}^2 - \frac{1}{2}\varphi_x u^2 \\
&\quad + \frac{1}{2}c_1\varphi_x^2 + \frac{1}{2}c_2\varphi_{xx}^2 \, dx \\
&= \int_{-L}^{L} \left(1+\frac{1}{2c_1}\right)\tilde{u}_x^2(\beta x) + \left(\frac{1}{2c_2}-1\right)\tilde{u}_{xx}^2(\beta x) - \frac{1}{2\gamma}\tilde{\varphi}_x(\beta x)\tilde{u}^2(\beta x) \\
&\quad + \frac{1}{2\gamma^2}c_1\tilde{\varphi}_x^2(\beta x) + \frac{1}{2\gamma^2}c_2\tilde{\varphi}_{xx}^2(\beta x)\, dx \\
&= \int_{-\beta L}^{\beta L} \left(1+\frac{1}{2c_1}\right)\beta\tilde{u}_{\tilde{x}}^2(\tilde{x}) + \left(\frac{1}{2c_2}-1\right)\beta^3\tilde{u}_{\tilde{x}\tilde{x}}^2(\tilde{x}) - \frac{1}{2\gamma}\tilde{\varphi}_{\tilde{x}}(\tilde{x})\tilde{u}^2(\tilde{x}) \\
&\quad + \frac{1}{2\gamma^2}c_1\beta\tilde{\varphi}_{\tilde{x}}^2(\tilde{x}) + \frac{1}{2\gamma^2}c_2\beta^3\tilde{\varphi}_{\tilde{x}\tilde{x}}^2(\tilde{x})\, d\tilde{x}
\end{aligned}
$$

so that taking $c_1 = \frac{1}{2}$, $c_2 = 1$, $\beta = 2$ and $\gamma = \frac{1}{8}$ we arrive at

$$
\frac{1}{4}\frac{d}{dt}\int_{-2L}^{2L} (\tilde{u}-8\tilde{\varphi})^2 d\tilde{x} \leq \int_{-2L}^{2L} 4\tilde{u}_{\tilde{x}}^2 - 4\tilde{u}_{\tilde{x}\tilde{x}}^2 - 4\tilde{\varphi}_{\tilde{x}}\tilde{u}^2 + 32\tilde{\varphi}_{\tilde{x}}^2 + 256\tilde{\varphi}_{\tilde{x}\tilde{x}}^2 \, d\tilde{x}.
$$

■

Combining the previous results we can now state that if there exists a potential φ such that the coercivity result claimed in (1.8) holds, then the solution tends towards the ball $B_r(0)$ with $r \lesssim \|\varphi\|_{H^2}$ and therefore can be estimated by this norm.

▶ **Lemma 2.3.**
Let $u(\cdot,t) \in H^2$ solve (KS), (PC_{2L}) and $\tilde{x}$ and $\tilde{u}$ be given by

$$
\begin{aligned}
\tilde{x} &= 2x, \\
\tilde{u}(x,t) &= u(2x,t).
\end{aligned}
$$

If there exists a function $\varphi \in H^2_{\text{per}}[-2L, 2L]$ with zero mean such that

$$
\int_{-2L}^{2L} \tilde{u}_{\tilde{x}\tilde{x}}^2 - \tilde{u}_{\tilde{x}}^2 + \varphi_{\tilde{x}}\tilde{u}^2 \, d\tilde{x} > \lambda_0\|\tilde{u}\|^2_{L^2} \tag{2.7}
$$

holds for some $\lambda_0 > 0$ independent of L, then there exists a constant $c = c(\lambda_0) > 0$, which is also independent of L, with

$$\limsup_{t\to\infty} \|u\|_{L^2[-L,L]} \leq c\|\varphi\|_{H^2[-2L,2L]}.$$

◀

Proof.
Lemma 2.2 implies

$$\frac{d}{dt}\int_{-2L}^{2L}(\tilde{u}-8\varphi)^2\,d\tilde{x} \leq -16\int_{-2L}^{2L}\tilde{u}_{\tilde{x}\tilde{x}}^2-\tilde{u}_{\tilde{x}}^2+\varphi_{\tilde{x}}\tilde{u}^2\,d\tilde{x}+16\int_{-2L}^{2L}8\varphi_{\tilde{x}}^2+64\varphi_{\tilde{x}\tilde{x}}^2\,d\tilde{x}$$
$$\overset{(2.7)}{<} -16\lambda_0\|\tilde{u}\|_{L^2}^2+128\|\varphi_{\tilde{x}}\|_{L^2}^2+1024\|\varphi_{\tilde{x}\tilde{x}}\|_{L^2}^2$$

such that Lemma 2.1 with $\tilde{L} = 2L$, $\bar{\lambda}_0 = 16\lambda_0$, $\tilde{\varphi} = 8\varphi$, $P^2 = 128\|\varphi_{\tilde{x}}\|_{L^2}^2 + 1024\|\varphi_{\tilde{x}\tilde{x}}\|_{L^2}^2$ yields

$$\operatorname{dist}(\tilde{u}(\cdot,t),B_r(0)) \leq e^{-4\lambda_0 t}\|\tilde{u}_0\|_{L^2}, \tag{2.8}$$

where $B_r(0)$ is the ball of radius

$$r = \sqrt{128\|\varphi\|_{L^2}^2+\frac{16}{\lambda_0}\left(\|\varphi_{\tilde{x}}\|_{L^2}^2+8\|\varphi_{\tilde{x}\tilde{x}}\|_{L^2}^2\right)}+16\|\varphi\|_{L^2}$$

in $L^2[-2L,2L]$ around 0. r is comparable to $\|\varphi\|_{H^2}$, i.e. there exist constants $c_1, c_2 > 0$ such that

$$c_1\|\varphi\|_{H^2[-2L,2L]} \leq r \leq c_2\|\varphi\|_{H^2[-2L,2L]}. \tag{2.9}$$

Since

$$\int_{-L}^{L}u^2(x,t)\,dx = \int_{-L}^{L}\tilde{u}^2(2x,t)\,dx = \frac{1}{2}\int_{-2L}^{2L}\tilde{u}^2(\tilde{x},t)\,d\tilde{x}$$

we get the desired inequality

$$\limsup_{t\to\infty}\|u(\cdot,t)\|_{L^2[-L,L]} = \frac{1}{2}\limsup_{t\to\infty}\|\tilde{u}(\cdot,t)\|_{L^2[-2L,2L]}$$

$$
\begin{aligned}
&\le \frac{1}{2}\limsup_{t\to\infty}\left(\inf_{\xi\in B_r(0)}(\|\tilde{u}(\cdot,t)-\xi\|_{L^2}+\|\xi\|_{L^2})\right)\\
&\le \frac{1}{2}r+\frac{1}{2}\limsup_{t\to\infty}\left(\inf_{\xi\in B_r(0)}\|\tilde{u}(\cdot,t)-\xi\|_{L^2}\right)\\
&\le \frac{1}{2}r+\frac{1}{2}\limsup_{t\to\infty}(\operatorname{dist}(\tilde{u}(\cdot,t),B_r(0)))\\
&\overset{(2.8)}{\le} \frac{1}{2}r+\frac{1}{2}\limsup_{t\to\infty} e^{-4\lambda_0 t}\|\tilde{u}_0\|_{L^2}\\
&= \frac{1}{2}r\\
&\overset{(2.9)}{\le} \frac{c_2}{2}\|\varphi\|_{H^2[-2L,2L]}.
\end{aligned}
$$

■

2.1.2 Construction of the Potential

Next we construct the $2L$-periodic potential function φ such that the coercivity condition (1.8) holds for all $\tilde{u}\in C^3[-L,L]$ with $\tilde{u}(0)=0$.

While technically, in order for Inequality (2.7) to hold, the construction would have to be done on $[-2L,2L]$ we will define it on $[-L,L]$ to omit unnecessary constants.

▶ **Lemma 2.4.**
If $u\in C^3[-a,a]$ with $u(0)=0$ and $v(y)=\frac{u(y)}{y}$, then

$$\int_{-a}^{a}\frac{1}{2}u_{yy}^2(y)\,dy \ge \int_{-a}^{a} v_y^2\,dy$$

and $v\in C^2[-a,a]$. ◀

Proof.
$u\in C^3[-a,a]$ and $u(0)=0$ imply $v\in C^2[-a,a]$, since

$$\frac{u(y)}{y}=\frac{\int_0^y u'(x)\,dx}{y}=\frac{\int_0^1 yu'(y\lambda)\,d\lambda}{y}=\int_0^1 u'(y\lambda)\,d\lambda,$$

and $u_{yy} = yv_{yy} + 2v_y$. Integration by parts yields

$$\begin{aligned}\int_{-a}^{a} yv_y v_{yy}\,dy &= -\int_{-a}^{a} v_y^2 + yv_y v_{yy}\,dy + yv_y^2\Big|_{-a}^{a} \\ &= -\int_{-a}^{a} v_y^2 + yv_y v_{yy}\,dy + a(v_y^2(a) + v_y^2(-a)) \\ &\geq -\int_{-a}^{a} v_y^2 + yv_y v_{yy}\,dy \end{aligned} \tag{2.10}$$

such that

$$\begin{aligned}\int_{-a}^{a} u_{yy}^2\,dy &= \int_{-a}^{a} \left(yv_{yy} + 2v_y\right)^2 dy \\ &= \int_{-a}^{a} y^2 v_{yy}^2 + 4yv_{yy}v_y + 4v_y^2\,dy \\ &\overset{(2.10)}{\geq} \int_{-a}^{a} y^2 v_{yy}^2 - 2v_y^2 + 4v_y^2\,dy \\ &\geq 2\int_{-a}^{a} v_y^2\,dy.\end{aligned}$$

■

Now we construct a piecewise constant function Q which can be arbitrary negative for small neighbourhoods of 0 while still leaving the form $\int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{2}v_y^2 + Q(y)v^2\,dy$ non-negative for all $v \in H^1[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$.

▶ **Lemma 2.5.**
Let the piecewise constant function Q be defined by

$$Q(y) = \begin{cases} -q_0 & \text{for } 0 \leq |y| \leq \frac{a}{2}, \\ q_1 & \text{for } \frac{a}{2} < |y| \leq a, \\ 0 & \text{for } a < |y|, \end{cases}$$

where $a, q_0, q_1 > 0$ satisfy

$$q_0 a^2 < 1, \qquad q_1 > \frac{q_0}{1 - a^2 q_0}, \qquad a < L^{\frac{4}{3}},$$

then

$$\int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{2}v_y^2 + Q(y)v^2\,dy \geq \int_{-a}^{a} \frac{1}{2}v_y^2 + Q(y)v^2\,dy \geq 0 \tag{2.11}$$

holds for all $v \in H^1[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$. ◀

Proof.
By definition of Q the first estimate in (2.11) is trivial. For the second estimate it suffices to show

$$\int_{0}^{a} \frac{1}{2}v_y^2 + Q(y)v^2\,dy \geq 0 \tag{2.12}$$

for all $v \in H^1[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$ since the negative part follows from Q being even and setting $\tilde{v}(y) = v(-y)$.

For $v \in H^1[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$, $y_1, y_2 \in [-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$ and $y_1 < y_2$ one gets

$$\begin{aligned}\int_{y_1}^{y_2} v_y^2\,dy &= \frac{1}{y_2 - y_1}\|1\|^2_{L^2[y_1,y_2]}\|v_y\|^2_{L^2[y_1,y_2]} \\ &\overset{(A.2)}{\geq} \frac{1}{y_2 - y_1}\|v_y\|^2_{L^1[y_1,y_2]} \\ &\geq \frac{1}{y_2 - y_1}\left(\int_{y_1}^{y_2} v_y\,dy\right)^2 \\ &= \frac{(v(y_2) - v(y_1))^2}{y_2 - y_1}. \end{aligned} \tag{2.13}$$

The Sobolev embedding (see Proposition A.7 in the Appendix) also holds in the special case $n = p = 1$ such that v is continuous and there exist points $y_0 = 0$, $y_1 \in [0, \frac{a}{2}]$ and $y_2 \in [\frac{a}{2}, a]$ with

$$\begin{aligned} v(y_0) &= v_0 = v(0), \\ v(y_1) &= v_1 = \max_{y \in [0, \frac{a}{2}]} |v(y)|, \\ v(y_2) &= v_2 = \min_{y \in [\frac{a}{2}, a]} |v(y)|. \end{aligned}$$

y_1 and y_2 are not necessary unique. In that case they can be chosen arbitrary in the set of points satisfying the maximum/minimum condition. These definitions yield

$$\begin{aligned}\int_0^a Q(y)v^2\,dy &= \int_0^{\frac{a}{2}} Q(y)v^2\,dy + \int_{\frac{a}{2}}^a Q(y)v^2\,dy \\ &= -q_0 \int_0^{\frac{a}{2}} v^2\,dy + q_1 \int_{\frac{a}{2}}^a v^2\,dy \\ &\geq \frac{-q_0 v_1^2 a}{2} + \frac{q_1 v_2^2 a}{2} \end{aligned} \tag{2.14}$$

and by the uncertainty estimate (2.13)

$$\begin{aligned}\int_0^a \frac{1}{2} v_y^2\,dy &\geq \int_0^{y_1} \frac{1}{2} v_y^2\,dy + \int_{y_1}^{y_2} \frac{1}{2} v_y^2\,dy \\ &\overset{(2.13)}{\geq} \frac{(v_1 - v_0)^2}{2y_1} + \frac{(v_2 - v_1)^2}{2(y_2 - y_1)} \\ &\geq \frac{(v_1 - v_0)^2}{a} + \frac{(v_2 - v_1)^2}{2a}. \end{aligned} \tag{2.15}$$

Combining the potential and kinetic estimates gives

$$\int_0^a \frac{1}{2} v_y^2 + Q(y)v^2\,dy \overset{(2.14),(2.15)}{\geq} \frac{(v_1 - v_0)^2}{a} + \frac{(v_2 - v_1)^2}{2a} + \frac{-q_0 v_1^2 a}{2} + \frac{q_1 v_2^2 a}{2}. \tag{2.16}$$

Writing v_0, v_1, v_2 as a vector

$$\tilde{v} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$$

and defining the quadratic matrix

$$A = \begin{pmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{3}{2a} - \frac{aq_0}{2} & -\frac{1}{2a} \\ 0 & -\frac{1}{2a} & \frac{1}{2a} + \frac{aq_1}{2} \end{pmatrix}$$

one gets

$$\begin{aligned} v^T A v &= \frac{v_0^2 - v_0 v_1}{a} - \frac{v_0 v_1}{a} + \frac{3v_1^2}{2a} - \frac{a q_0 v_1^2}{2} - \frac{v_1 v_2}{2a} - \frac{v_1 v_2}{2a} + \frac{v_2^2}{2a} + \frac{a q_1 v_2^2}{2} \\ &= \frac{(v_1 - v_0)^2}{a} + \frac{(v_2 - v_1)^2}{2a} + \frac{-q_0 v_1^2 a}{2} + \frac{q_1 v_2^2 a}{2} \\ &\overset{(2.16)}{\leq} \int_0^a \frac{1}{2} v_y^2 + Q(y) v^2 \, dy. \end{aligned} \tag{2.17}$$

This quadratic form is positive definite since the principle minors of A, given by

$$\det\left(\tfrac{1}{a}\right) = \frac{1}{a},$$

$$\det\begin{pmatrix} \frac{1}{a} & -\frac{1}{a} \\ -\frac{1}{a} & \frac{3}{2a} - \frac{a q_0}{2} \end{pmatrix} = \frac{1}{2}\left(\frac{1}{a^2} - q_0\right),$$

$$\det\begin{pmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{3}{2a} - \frac{a q_0}{2} & -\frac{1}{2a} \\ 0 & -\frac{1}{2a} & \frac{1}{2a} + \frac{a q_1}{2} \end{pmatrix} = \frac{1}{4a}\left(-q_0 + q_1\left(1 - a^2 q_0\right)\right),$$

are positive by assumption. Therefore the left-hand side of (2.17) is non negative for all $v \in H^1[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$, showing (2.12) and therefore concluding the proof. ■

The assumptions on a, q_0 and q_1 are physically justified in the sense that we are investigating the asymptotic behaviour where L grows, meaning the constant a will be smaller than $L^{\frac{4}{3}}$ for $L \to \infty$. $q_0 < \frac{1}{a^2}$ sets a minimum of the potential in comparison to the width of the domain, which would otherwise lead to a potential valley resulting in a bounded state. The condition $q_1 > \frac{q_0}{1-a^2 q_0}$, characterizing a minimum on the positive part of the potential, sets a minimal bound of $\frac{a^2 q_0}{2(1-a^2 q_0^2)}$ on the mean of Q therefore requiring a non-strictly negative potential.

Clearly for positive a one always finds positive q_0 and q_1 satisfying these conditions.

Now we smoothen the previously defined function Q and through dividing by y^2 we emphasize the negative part of Q such that the average of this new function can be made arbitrarily small while not changing the positivity of the previous form.

▶ **Lemma 2.6.**
Let q_0 and q_1 be defined as in Lemma 2.5. For every $\bar{c} > 0$ there exists a function $\tilde{Q}$, bounded by $-q_0 \leq \tilde{Q} \leq q_1$, such that

1. $\tilde{q} = \frac{\tilde{Q}(y)}{y^2} \in C_c^\infty[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$,
2. $\int \tilde{q}\, dy \leq -\bar{c}$,
3. $\int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{2}v_y^2 + \tilde{Q}v^2 \geq 0$ for all $v \in H^1[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$. ◀

Proof.
Let Q be defined as in Lemma 2.5 and $g \in C_0^\infty$ as

$$g(y) = \begin{cases} e^{-\left(\frac{1}{y^2}+\frac{1}{(y-1)^2}\right)} & \text{for } y \in (0,1), \\ 0 & \text{for } y \notin (0,1). \end{cases}$$

Setting

$$f(y) = \frac{\int_0^y g(s)\, ds}{\int_0^1 g(s)\, ds}$$

gives a non decreasing C^∞ function with $f(y) = 0$ for $y \leq 0$ and $f(y) = 1$ for $y \geq 1$. Via f we can construct a smooth version of Q by

$$\tilde{Q}(y) = \begin{cases} -q_0 f\left(\frac{y}{\delta}\right) & \text{for } |y| \in [0,\delta), \\ -q_0 & \text{for } |y| \in \left[\delta, \frac{a}{2}-\delta\right), \\ -q_0 + (q_0+q_1) f\left(\frac{y-\frac{a}{2}+\delta}{\delta}\right) & \text{for } |y| \in \left[\frac{a}{2}-\delta, \frac{a}{2}\right), \\ q_1 & \text{for } |y| \in \left[\frac{a}{2}, a\right), \\ q_1 f\left(\frac{\delta+a-y}{\delta}\right) & \text{for } |y| \in [a, a+\delta), \\ 0 & \text{for } |y| \in [a+\delta, \infty), \end{cases}$$

where

$$\delta < \frac{a}{4} \tag{2.18}$$

in order for $\tilde{Q}$ to be well-defined. $\tilde{Q} \in C_c^\infty$, $\tilde{Q}(0) = 0$ implies $\tilde{q} = \frac{\tilde{Q}}{y^2} \in C_c^\infty$ with $\tilde{q}(0) = 0$ and since $0 \leq f \leq 1$ yields $-q_0 \leq \tilde{Q} \leq q_1$ one gets

$$\begin{aligned}\int_{\mathbb{R}} \tilde{q}(y)\, dy &\leq 2\left(-q_0 \int_\delta^{\frac{a}{2}-\delta} \frac{1}{y^2}\, dy + q_1 \int_{\frac{a}{2}-\delta}^{a+\delta} \frac{1}{y^2}\, dy\right) \\ &= -2q_0\left(\frac{1}{\delta} - \frac{1}{\frac{a}{2}-\delta}\right) + 2q_1 \frac{\frac{a}{2}+2\delta}{\left(\frac{a}{2}-\delta\right)(a+\delta)} \\ &\overset{(2.18)}{\leq} -\frac{2}{\delta} q_0 + \frac{8}{a}(q_0+q_1)\end{aligned}$$

such that for fixed a the mean of $\tilde{q}$ can be made arbitrary negative by choosing δ sufficiently small, which proves part 2. For sufficiently small δ one has $(-(a+\delta), a+\delta) \subset (-L^{\frac{4}{3}}, L^{\frac{4}{3}})$ since a is strictly smaller than $L^{\frac{4}{3}}$, which together with $\tilde{Q} = 0$ for $|y| \geq a+\delta$ and $\tilde{Q} \geq Q$ yields

$$\int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{2}v_y^2 + \tilde{Q}v^2 = \int_{-(a+\delta)}^{a+\delta} \frac{1}{2}v_y^2 + \tilde{Q}v^2 \geq \int_{-a}^{a} \frac{1}{2}v_y^2 + Qv^2 \overset{(2.11)}{\geq} 0$$

by Lemma 2.5 proving part 3. ■

We now define q as a rescaling of $\tilde{q}$ and construct the final potential through $\varphi_x = q(x) - \langle q \rangle$, where $\langle \cdot \rangle$ is the average on $[-L, L]$ in order for its derivative to have zero mean.

▶ **Theorem 2.7.**
There exists a $2L$-periodic function φ_x with zero mean such that

$$\int_{-L}^{L} u_{xx}^2 - u_x^2 + \varphi_x u^2\, dx \geq \frac{1}{4}\int_{-L}^{L} u_{xx}^2 + u^2\, dx \geq \frac{1}{4}\|u\|_{L^2}^2$$

holds for every $2L$-periodic $u \in C^3[-L, L]$ with $u(0) = 0$. ◀

Proof.
Let $\tilde{q}$ and $\tilde{Q}$ be defined by Lemma 2.6 with $\bar{c} = \frac{3}{2}$ such that

$$\int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \tilde{q}\, dy \leq -\bar{c} = -\frac{3}{2}. \tag{2.19}$$

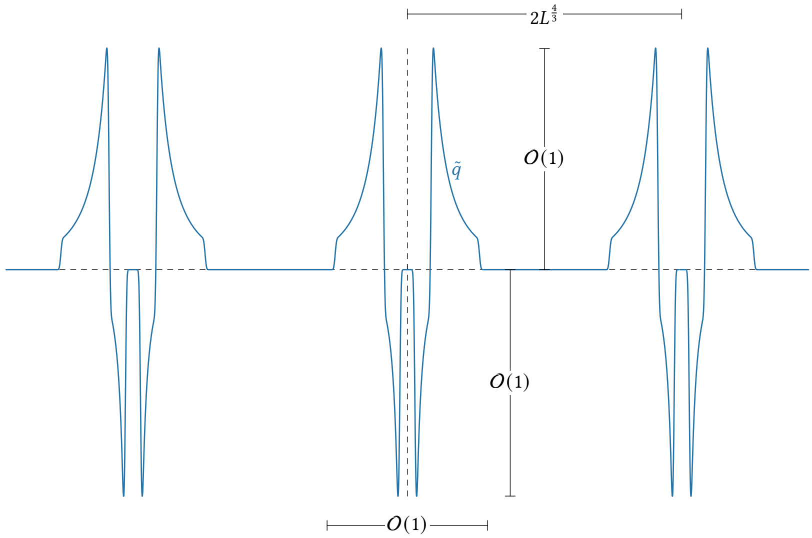

Figure 2.1: The function $\tilde{q}$ constructed in Lemma 2.6 that, except for its periodicity, is independent of L.

Then part 3 of Lemma 2.6 yields

$$0 \leq \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{2} v_y^2 + \tilde{Q} v^2 \, dy \tag{2.20}$$

for every $v \in H^1[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$. Define the rescaling

$$\begin{aligned} \tilde{u}\left(L^{\frac{1}{3}} x\right) &= u(x), \\ y &= L^{\frac{1}{3}} x, \end{aligned} \tag{2.21}$$

which implies $\tilde{u} \in C^3[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$ with $\tilde{u}(0) = 0$. By Lemma 2.4 $v(y) := \frac{\tilde{u}(y)}{y} \in H^1[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$ and

$$\int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} v_y^2 \, dy \leq \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{2} \tilde{u}_{yy}^2(y) \, dy \tag{2.22}$$

such that for

$$q(x) = L^{\frac{4}{3}} \tilde{q}(L^{\frac{1}{3}} x)$$

one gets

$$\begin{aligned}
0 &\overset{(2.20)}{\leq} L \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{2} v_y^2 + \tilde{Q} v^2 \, dy \\
&\overset{(2.22)}{\leq} L \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{4} \tilde{u}_{yy}^2 + \tilde{Q} v^2 \, dy \\
&= L \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{4} \tilde{u}_{yy}^2(y) + \tilde{q}(y) \tilde{u}^2(y) \, dy \\
&\overset{(2.21)}{=} L \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \frac{1}{4} u_{yy}^2 \left(L^{-\frac{1}{3}} y\right) + L^{-\frac{4}{3}} q\left(L^{-\frac{1}{3}} y\right) u^2 \left(L^{-\frac{1}{3}} y\right) \, dy \\
&= \int_{-L}^{L} \frac{1}{4} u_{xx}^2 + q u^2 \, dx. \qquad (2.23)
\end{aligned}$$

Defining

$$\varphi_x(x) = q(x) - \langle q \rangle,$$

where

$$\langle q \rangle = \frac{1}{2L} \int_{-L}^{L} q(x) \, dx = \frac{1}{2} L^{\frac{1}{3}} \int_{-L}^{L} \tilde{q}(L^{\frac{1}{3}} x) \, dx = \frac{1}{2} \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \tilde{q}(y) \, dy \overset{(2.19)}{\leq} -\frac{3}{4} \qquad (2.24)$$

is the mean value of q on $[-L, L]$, yields

$$\begin{aligned}
0 &\overset{(2.23)}{\leq} \int_{-L}^{L} \frac{1}{4} u_{xx}^2 + q u^2 \, dx \\
&= \int_{-L}^{L} \frac{1}{4} u_{xx}^2 + (\varphi_x + \langle q \rangle) u^2 \, dx \\
&\overset{(2.24)}{\leq} \int_{-L}^{L} \frac{1}{4} u_{xx}^2 + \left(\varphi_x - \frac{3}{4}\right) u^2 \, dx. \qquad (2.25)
\end{aligned}$$

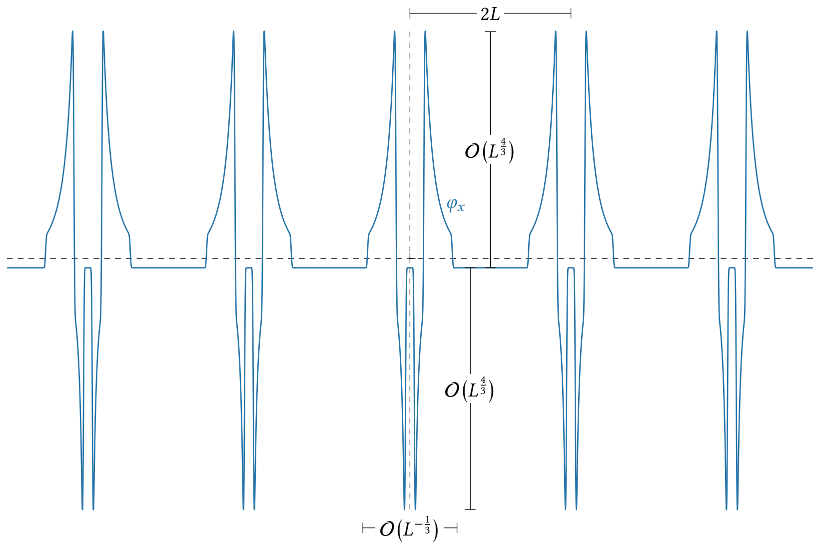

Figure 2.2: The function φ_x constructed in Theorem 2.7.

Partial integration, the periodicity of u and Cauchy inequality (see Proposition A.3 in the Appendix) imply

$$\int_{-L}^{L} u_x^2 \, dx = -\int_{-L}^{L} u u_{xx} \, dx \overset{(A.3)}{\leq} \frac{1}{2}\int_{-L}^{L} u^2 \, dx + \frac{1}{2}\int_{-L}^{L} u_{xx}^2 \, dx \tag{2.26}$$

and therefore

$$\begin{aligned}\int_{-L}^{L} (u_{xx}^2 - u_x^2 + \varphi_x u^2) dx &\overset{(2.26)}{\geq} \int_{-L}^{L} \frac{1}{2}u_{xx}^2 + \left(\varphi_x - \frac{1}{2}\right) u^2 \, dx \\ &\overset{(2.25)}{\geq} \int_{-L}^{L} \frac{1}{2}u_{xx}^2 - \frac{1}{2}u^2 - \frac{1}{4}u_{xx}^2 + \frac{3}{4}u^2 \, dx \\ &= \frac{1}{4}\int_{-L}^{L} u_{xx}^2 + u^2 \, dx.\end{aligned}$$

■

Since the potential is only defined up to a constant we can set $\varphi(0) = 0$ such that $\|\varphi\|_{H^2}$ can be bounded by $cL^{\frac{3}{2}}$.

▶ **Corollary 2.8.**
For the periodic function φ constructed in Theorem 2.7 there exists a constant $c > 0$ such that

$$\|\varphi\|_{H^2} \leq cL^{\frac{3}{2}} \tag{2.27}$$

holds. ◀

Proof.
Cauchy inequality (see Proposition A.3 in the Appendix) gives

$$(q(x) - \langle q \rangle)^2 = q^2(x) - 2q(x)\langle q \rangle + \langle q \rangle^2 \overset{(A.3)}{\leq} 2\big(q^2(x) + \langle q \rangle^2\big) \tag{2.28}$$

such that via Jensen inequality (see Proposition A.4 in the Appendix)

$$\begin{aligned}
\|\varphi_x\|^2_{L^2(-L,L)} &= \int_{-L}^{L} (q(x) - \langle q \rangle)^2 dx \\
&\overset{(2.28)}{\leq} 2\int_{-L}^{L} q^2(x)\, dx + 2\int_{-L}^{L} \left(\frac{1}{2L}\int_{-L}^{L} q(y)\, dy\right)^2 dx \\
&\overset{(A.4)}{\leq} 2\int_{-L}^{L} q^2(x)\, dx + 2\int_{-L}^{L} \frac{1}{2L}\int_{-L}^{L} q^2(y)\, dy\, dx \\
&= 4\int_{-L}^{L} q^2(x)\, dx.
\end{aligned} \tag{2.29}$$

By definition $q(x) = L^{\frac{4}{3}}\tilde{q}(L^{\frac{1}{3}}x)$, where $\tilde{q} \in C_c^\infty[-L^{\frac{4}{3}}, L^{\frac{4}{3}}]$ does not depend on L, yielding

$$\begin{aligned}
\|\varphi_x\|^2_{L^2(-L,L)} &\overset{(2.29)}{\leq} 4\int_{-L}^{L} q^2(x)\, dx \\
&= 4L^{\frac{8}{3}}\int_{-L}^{L} \tilde{q}^2(L^{\frac{1}{3}}x)\, dx \\
&= 4L^{\frac{8}{3}}\int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \tilde{q}^2(\lambda)L^{-\frac{1}{3}}\, d\lambda \\
&= 4L^{\frac{7}{3}}\int_{\operatorname{supp}(\tilde{q})} \tilde{q}^2\, d\lambda
\end{aligned}$$

$$\leq cL^{\frac{7}{3}}$$

and similarly

$$\begin{aligned}
\|\varphi_{xx}\|^2_{L^2(-L,L)} &= \int_{-L}^{L} q_x^2(x)\,dx \\
&= L^{\frac{8}{3}} \int_{-L}^{L} \tilde{q}_x^2(L^{\frac{1}{3}}x)\,dx \\
&= L^{\frac{10}{3}} \int_{-L}^{L} \left(\tilde{q}'(L^{\frac{1}{3}}x)\right)^2 dx \\
&= L^3 \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \tilde{q}_\lambda^2(\lambda)\,d\lambda \\
&\leq cL^3.
\end{aligned}$$

φ is only defined up to a constant, so let $\varphi(0) = 0$ such that

$$\begin{aligned}
\varphi(x) &= \int_0^x \varphi_s\,ds \\
&= \int_0^x q(s) - \langle q\rangle\,ds \\
&= L^{\frac{4}{3}} \int_0^x \tilde{q}(L^{\frac{1}{3}}s)ds + \frac{1}{2L}x \int_{-L}^{L} q(s)\,ds \\
&= L \int_0^{L^{\frac{1}{3}}x} \tilde{q}(y)\,dy + \frac{1}{2}L^{\frac{1}{3}}x \int_{-L}^{L} \tilde{q}(L^{\frac{1}{3}}s)\,ds \\
&= L \int_0^{L^{\frac{1}{3}}x} \tilde{q}(y)\,dy + \frac{1}{2}x \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} \tilde{q}(\lambda)\,d\lambda.
\end{aligned} \tag{2.30}$$

$\tilde{q}$ is independent of L and compactly supported implying

$$\begin{aligned}
\|\varphi\|_{L^\infty} &\leq L\left(1 + \frac{1}{2}\right) \int_{-L^{\frac{4}{3}}}^{L^{\frac{4}{3}}} |\tilde{q}(y)|\,dy \\
&= L\left(1 + \frac{1}{2}\right) \int_{\mathrm{supp}(\tilde{q})} |\tilde{q}(y)|\,dy \\
&\leq \tilde{c}L,
\end{aligned}$$

which in return yields

$$\|\varphi\|^2_{L^2(-L,L)} = \int_{-L}^{L} \varphi^2(x)\, dx \leq 2L\|\varphi\|^2_{L^\infty} \leq cL^3.$$

■

2.1.3 Bound for the Kuramoto-Sivashinsky Equation

By [Tem97, p.143 and Theorem III.3.1] the solution u of the initial boundary value problem (KS), (PC_{2L}), (BC_0), (IC) has regularity $u(\cdot, t) \in H^4$ for all times $t > 0$. Therefore Sobolev embedding (see Proposition A.7 in the Appendix) yields $u(\cdot, t) \in C^3[-L, L]$.

Using the potential function constructed in Section 2.1.2 we can now apply the argument stated in Section 2.1.1 and derive bounds for the solution of the Kuramoto-Sivashinsky equation.

► **Corollary 2.9 (Energy Bound).**
Let u solve (KS), (PC_{2L}), (BC_0), (IC), then there exists a universal constant c such that

$$\limsup_{t\to\infty} \|u\|_{L^2[-L,L]} \leq cL^{\frac{3}{2}}.$$

◄

Proof.
By Theorem 2.7 there exists a $2\tilde{L}$-periodic function φ_x such that

$$\int_{-\tilde{L}}^{\tilde{L}} \tilde{u}^2_{xx} - \tilde{u}^2_x + \varphi_{\tilde{x}}\tilde{u}^2\, d\tilde{x} > \frac{1}{5}\|\tilde{u}\|^2_{L^2}$$

holds for every $0 \not\equiv \tilde{u} \in C^3[-\tilde{L}, \tilde{L}]$ with $u(0) = 0$ and $2\tilde{L}$-periodicity. Setting $\tilde{u}(x, t) = u(2x, t)$ and $\tilde{L} = 2L$ in order to match the scaling in Lemma 2.3, then for $\lambda_0 = \frac{1}{5}$ this implies

$$\limsup_{t\to\infty} \|u\|_{L^2[-L,L]} \leq c_1\|\varphi\|_{H^2[-2L,2L]}$$

and by Corollary 2.8

$$\limsup_{t\to\infty} \|u\|_{L^2[-L,L]} \leq c_1 \|\varphi\|_{H^2[-2L,2L]} \overset{(2.27)}{\leq} c_1 c_2 (2L)^{\frac{3}{2}} = cL^{\frac{3}{2}}.$$

■

2.2 Viewing Kuramoto-Sivashinsky as a Perturbation of Burgers' Equation

Based on [GO05] in this Section we prove the statements outlined in Section 1.2 of Chapter 1. First we introduce basic properties of the Kuramoto-Sivashinsky equation in Section 2.2.1, afterwards, in Section 2.2.2, the convergence of rescaled solutions of (KS) to solutions of (BE) with entropy condition ($EC_{\hat{u}^2}$) is shown. Based on [DOW04] in Section 2.2.3 the justification why the right-hand side of this entropy condition is negligible is proven so that the rescaled solution of the Kuramoto-Sivashinsky equation solves (BE) with entropy condition (EC_0). Afterwards, in Section 2.2.4, we derive the bound claimed in Inequality (1.14) for such solutions. In Section 2.2.5 we translate this bound back to the original solution of the initial boundary value problem (KS), (PC_L), (IC).

2.2.1 Properties of the Kuramoto-Sivashinsky Equation

First we prove translation invariance of (KS) in space and time.

▶ **Lemma 2.10 (Translation Invariance).**
The Kuramoto Sivashinsky equation is translation invariant in space and time, i.e. if $u(x,t)$ is a solution of (KS), then so is $v(x,t) = u(x+y, t+\tau)$. ◀

Proof.
Let $z := x + y$ and $\lambda := t + \tau$, then by chain rule $u_t(x+y, t+\tau) = u_\lambda(z,\lambda)$ and $u_x(x+y,t+\tau) = u_z(z,\lambda)$, so that

$$\begin{aligned}
&v_t(x,t) + \left(\frac{1}{2}v^2(x,t)\right)_x + v_{xx}(x,t) + v_{xxxx}(x,t)\\
&= u_t(x+y,t+\tau) + \left(\frac{1}{2}u^2(x+y,t+\tau)\right)_x\\
&\qquad + u_{xx}(x+y,t+\tau) + u_{xxxx}(x+y,t+\tau)\\
&= u_\lambda(z,\lambda) + \left(\frac{1}{2}u^2(z,\lambda)\right)_z\\
&\qquad + u_{zz}(z,\lambda) + u_{zzzz}(z,\lambda)\\
&= 0
\end{aligned}$$

and the initial value is given by $v_0(x) = v(x,0) = u(x+y,\tau)$. ■

Next we derive energy estimates for solutions of the initial boundary value problem.

► **Lemma 2.11 (Energy Estimates).**
Let u be a solution of the Kuramoto-Sivashinsky equation (KS) with L-periodicity ($\mathrm{PC_L}$) and initial values that fulfill (IC), then the estimates

$$\sup_{t\in(s,s+T)} \int u^2(t)\,dx \lesssim e^{\frac{T}{2}} \int u^2(s)\,dx, \tag{2.31}$$

$$\int_s^{s+T} \int u_x^2\,dxdt \lesssim e^{\frac{T}{2}} \int u^2(s)\,dx, \tag{2.32}$$

$$\int_s^{s+T} \int u_{xx}^2\,dxdt \lesssim e^{\frac{T}{2}} \int u^2(s)\,dx, \tag{2.33}$$

$$\left(\int_s^{s+T} \int |u_x|^3\,dxdt\right)^{\frac{2}{3}} \lesssim e^{\frac{T}{2}} \int u^2(s)\,dx \tag{2.34}$$

hold for all $s \geq 0$ and $T > 0$. ◄

Proof.
Because of translation invariance in time (see Lemma 2.10) we can assume $s = 0$. The calculation

$$\int u^2 u_x\,dx = -\int \left(u^2\right)_x u\,dx = -2\int u^2 u_x\,dx$$

implies

$$\int u^2 u_x\,dx = 0. \tag{2.35}$$

With this we get

$$\begin{aligned}
\frac{d}{dt}\int \frac{1}{2}u^2\,dx &= \int u u_t\,dx \\
&\overset{\text{(KS)}}{=} \int u\left(-\left(\frac{u^2}{2}\right)_x - u_{xx} - u_{xxxx}\right)dx \\
&= -\int u^2 u_x\,dx - \int (u u_{xx} + u u_{xxxx})dx
\end{aligned}$$

$$\overset{(2.35)}{=} \int u_x^2 - u_{xx}^2 \, dx. \tag{2.36}$$

Via Hölder and Cauchy inequality (see Propositions A.2 and A.3 in the Appendix) the first integrand can be estimated by

$$\begin{aligned} \int u_x^2 \, dx &= - \int u u_{xx} \, dx \\ &\leq \|u u_{xx}\|_{L^1([0,L])} \\ &\overset{(A.2)}{\leq} \|u\|_{L^2([0,L])} \|u_{xx}\|_{L^2([0,L])} \\ &\overset{(A.3)}{\leq} \frac{1}{2c} \|u\|^2_{L^2([0,L])} + \frac{c}{2} \|u_{xx}\|^2_{L^2([0,L])} \end{aligned} \tag{2.37}$$

for all $c > 0$. Using Estimate (2.37) with $c = 2$ in Equality (2.36) one gets

$$\begin{aligned} \frac{d}{dt} \int \frac{1}{2} u^2 \, dx &= \int u_x^2 - u_{xx}^2 \, dx \\ &\overset{(2.37)}{\leq} \frac{1}{4} \|u\|^2_{L^2([0,L])} + \|u_{xx}\|^2_{L^2([0,L])} - \int u_{xx}^2 \, dx \\ &= \frac{1}{2} \int \frac{1}{2} u^2 \, dx \end{aligned}$$

and by Grönwall inequality (see Proposition A.9 in the Appendix)

$$\int u^2(t) \, dx \leq e^{\frac{t}{2}} \int u_0^2 \, dx, \tag{2.38}$$

which implies (2.31). Integrating Equation (2.36) we get

$$\frac{1}{2} \int u^2(t) \, dx = \frac{1}{2} \int u_0^2 \, dx + \int_0^t \int u_x \, dxdt - \int_0^t \int u_{xx} \, dxdt$$

so that setting $c = 1$ in Inequality (2.37) we have

$$\frac{1}{2} \int u^2(t) \, dx + \int_0^t \int u_{xx} \, dxdt = \frac{1}{2} \int u_0^2 \, dx + \int_0^t \int u_x \, dxdt$$

$$\overset{(2.37)}{\leq} \frac{1}{2}\int u_0^2\,dx + \frac{1}{2}\int_0^t \int u_{xx}^2\,dxdt + \frac{1}{2}\int_0^t \int u^2(t)\,dxdt,$$

which together with Estimate (2.38) implies

$$\begin{aligned}
\int_0^t \int u_{xx}\,dxdt &\leq \int u_0^2\,dx - \int u^2(t)\,dx + \int_0^t \int u^2\,dxdt \\
&\leq \int u_0^2\,dx + \int_0^t \int u^2\,dxdt \\
&\overset{(2.38)}{\lesssim} \int u_0^2\,dx + \int_0^T e^{\frac{t}{2}} \int u_0^2\,dxdt \\
&= \left(2e^{\frac{T}{2}} - 1\right) \int u_0^2\,dx \\
&\lesssim e^{\frac{T}{2}} \int u_0^2\,dx,
\end{aligned}$$

proving Inequality (2.33). Estimate (2.32) follows from Inequalities (2.31), (2.33) and (2.37) with $c = 1$, since in fact

$$\begin{aligned}
\int_0^T \int u_x^2\,dxdt &\overset{(2.37)}{\leq} \frac{1}{2}\int_0^T \int u^2\,dxdt + \frac{1}{2}\int_0^T \int u_{xx}^2 \\
&\overset{(2.33)}{\lesssim} \frac{1}{2}\sup_{t\in(0,T}\int u^2(t)\,dx \int_0^T dt + e^{\frac{T}{2}} \int u_0^2\,dx \\
&\overset{(2.31)}{\lesssim} e^{\frac{T}{2}} \int u_0^2\,dx(T+1) \\
&\lesssim e^{\frac{T}{2}} \int u_0^2\,dx.
\end{aligned}$$

By Inequality (2.33) $u_{xx} \in L^2([0,L])$ so that the Sobolev embedding (see Proposition A.7 in the Appendix) gives

$$u_x \in C^{0,\frac{1}{2}}([0,L]) \tag{2.39}$$

and because of the periodicity condition (PC_L), there exists a $y \in [0, L]$ such that

$$u_x(y) = 0. \tag{2.40}$$

This yields

$$\begin{aligned}
u_x^2(x) &\overset{(2.40)}{=} u_x^2(x) - u_x^2(y) \\
&= \int_y^x \partial_x u_x^2 \, dx \\
&\leq \int \left|(u_x^2)_x\right| dx \\
&= 2\|u_x u_{xx}\|_{L^1([0,L])} \\
&\overset{(A.2)}{\lesssim} \|u_x\|_{L^2([0,L])} \|u_{xx}\|_{L^2([0,L])}
\end{aligned}$$

and therefore

$$\sup_{x\in[0,L]} |u_x| \lesssim \|u_x\|_{L^2([0,L])}^{\frac{1}{2}} \|u_{xx}\|_{L^2([0,L])}^{\frac{1}{2}}. \tag{2.41}$$

Using (2.41) together with Inequalities (2.37), (2.31), (2.32), (2.33) we arrive at

$$\begin{aligned}
&\int_0^T \int |u_x|^3 \, dxdt \\
&\overset{(2.41)}{\lesssim} \int_0^T \|u_x\|_{L^2([0,L])}^{\frac{1}{2}} \|u_{xx}\|_{L^2([0,L])}^{\frac{1}{2}} \int |u_x|^2 \, dxdt \\
&= \int_0^T \left(\int u_x^2 \, dx\right)^{1+\frac{1}{4}} \left(\int u_{xx}^2 \, dx\right)^{\frac{1}{4}} dt \\
&\overset{(2.37)}{\leq} \int_0^T \left(\int u^2 \, dx\right)^{\frac{1}{2}} \left(\int u_{xx}^2 \, dx\right)^{\frac{1}{2}} \left(\int u_x^2 \, dx\right)^{\frac{1}{4}} \left(\int u_{xx}^2 \, dx\right)^{\frac{1}{4}} dt \\
&\leq \sup_{t\in(0,T)} \left(\int u^2(t) \, dx\right)^{\frac{1}{2}} \int_0^T \left(\int u_x^2 \, dx\right)^{\frac{1}{4}} \left(\int u_{xx}^2 \, dx\right)^{\frac{3}{4}} dt \\
&\overset{(2.31)}{\lesssim} \left(e^{\frac{T}{2}} \int u_0^2 \, dx\right)^{\frac{1}{2}} \int_0^T \left(\int u_x^2 \, dx\right)^{\frac{1}{4}} \left(\int u_{xx}^2 \, dx\right)^{\frac{3}{4}} dt
\end{aligned}$$

$$\overset{(A.2)}{\leq} \left(e^{\frac{T}{2}} \int u_0^2 \, dx\right)^{\frac{1}{2}} \left(\int_0^T \int u_x^2 \, dxdt\right)^{\frac{1}{4}} \left(\int_0^T \int u_{xx}^2 \, dxdt\right)^{\frac{3}{4}}$$
$$\overset{(2.32),(2.33)}{\lesssim} \left(e^{\frac{T}{2}} \int u_0^2 \, dx\right)^{\frac{3}{2}},$$

which concludes the proof of Estimate (2.34). ■

Using the connection between conservation laws and Hamilton-Jacobi equations described in Section 1.2 of Chapter 1 we now define h and get integral identities for u and h, which will be needed in Lemma 2.13 and Theorem 2.16.

▶ **Lemma 2.12.**
For a L-periodic (PC_L) solution u of the Kuramoto-Sivashinsky equation (KS) with initial condition (IC) and h such that

1.
$$\begin{pmatrix} h_t \\ h_x \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{2}u^2 + u_x + u_{xxx}\right) + \frac{1}{L}\int \frac{1}{2}u^2 \, dx \\ u \end{pmatrix}, \tag{2.42}$$

one has

$$\frac{1}{12}\int_0^\infty \int u^4 \, dx \, \zeta \, dt + \frac{1}{L}\int_0^\infty \left(\int \frac{1}{2}u^2 \, dx\right)^2 \zeta \, dt + \int_0^\infty \int u_x^3 \, dx \, \zeta \, dt$$
$$= -\int_0^\infty \int (u_x^2 - u_{xx}^2) h \, dx \, \zeta \, dt - \int_0^\infty \int \frac{1}{2}u^2 h \, dx \, \zeta_t \, dt$$

2.
$$\begin{pmatrix} h_t \\ h_x \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{2}u^2 + u_x + u_{xxx}\right) \\ u \end{pmatrix}, \tag{2.43}$$

one has

$$\frac{1}{12}\int_0^\infty \int u^4 \, dx \, \zeta \, dt + \int_0^\infty \int u_x^3 \, dx \, \zeta \, dt + \int_0^\infty \int \frac{1}{4}u^2 h \, dx \, \zeta \, dt$$
$$= \int_0^\infty \int \left(\frac{1}{2}u + u_{xx}\right)^2 h \, dx \, \zeta \, dt - \int_0^\infty \int \frac{1}{2}u^2 h \, dx \, \zeta_t \, dt \tag{2.44}$$

for all $\zeta \in C_c^\infty((0,\infty))$. ◀

Proof.
As discussed in Section 1.2 of Chapter 1 there exists h such that

$$\begin{pmatrix} h_t \\ h_x \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{2}u^2 + u_x + u_{xxx}\right) + g \\ u \end{pmatrix}, \tag{2.45}$$

where g is spatially constant in order to prove both statements at once. We define

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} := \begin{pmatrix} \frac{1}{2}u^2 \\ \frac{1}{3}u^3 + uu_x + uu_{xxx} - u_x u_{xx} \end{pmatrix}, \tag{2.46}$$

which like h has L-periodicity and calculate

$$\begin{aligned}
\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} \cdot V &= \left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3 + uu_x + uu_{xxx} - u_x u_{xx}\right)_x \\
&= uu_t + u^2 u_x + u_x^2 + uu_{xx} + uu_{xxxx} - u_{xx}^2 \\
&\overset{\text{(KS)}}{=} -u^2 u_x - uu_{xx} - uu_{xxxx} + u^2 u_x + u_x^2 + uu_{xx} + uu_{xxxx} - u_{xx}^2 \\
&= u_x^2 - u_{xx}^2
\end{aligned} \tag{2.47}$$

such that partial integration yields

$$\begin{aligned}
&\int_0^\infty \int V \cdot \begin{pmatrix} h_t \\ h_x \end{pmatrix} dx\, \zeta\, dt \\
&\quad = -\int_0^\infty \int h \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} \cdot V\, dx\, \zeta\, dt - \int_0^\infty \int hV_1\, dx\, \zeta_t\, dt \\
&\quad \overset{(2.47)}{=} -\int_0^\infty \int h\left(u_x^2 - u_{xx}^2\right) dx\, \zeta\, dt - \int_0^\infty \int hV_1\, dx\, \zeta_t\, dt \\
&\quad \overset{(2.46)}{=} -\int_0^\infty \int h\left(u_x^2 - u_{xx}^2\right) dx\, \zeta\, dt - \int_0^\infty \frac{1}{2}u^2 h\, dx\, \zeta_t\, dt.
\end{aligned} \tag{2.48}$$

Looking at the left-hand side of this equation we get, by the Definitions (2.45) of h and (2.46) of V,

$$
\begin{aligned}
\int_0^\infty \int V \cdot \begin{pmatrix} h_t \\ h_x \end{pmatrix} dx\, \zeta\, dt
&\overset{(2.45),(2.46)}{=} \int_0^\infty \int \begin{pmatrix} \frac{1}{2}u^2 \\ \frac{1}{3}u^3 + uu_x + uu_{xxx} - u_x u_{xx} \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} -\left(\frac{1}{2}u^2 + u_x + u_{xxx}\right) + g \\ u \end{pmatrix} dx\, \zeta\, dt \\
&= \int_0^\infty \int \Big(-\frac{1}{4}u^4 - \frac{1}{2}u^2 u_x - \frac{1}{2}u^2 u_{xxx} + \frac{1}{2}u^2 g + \frac{1}{3}u^4 \\
&\qquad + u^2 u_x + u^2 u_{xxx} - uu_x u_{xx} \Big) dx\, \zeta\, dt \\
&= \int_0^\infty \int \left(\frac{1}{12}u^4 + \frac{1}{2}u^2 g \right) dx\, \zeta\, dt \\
&\qquad + \int_0^\infty \int \left(\frac{1}{2}u^2 u_x + \frac{1}{2}u^2 u_{xxx} - uu_x u_{xx} \right) dx\, \zeta\, dt.
\end{aligned} \tag{2.49}
$$

Partial integration implies

$$
\int uu_x u_{xx}\, dx = -\int u_x^3\, dx - \int uu_x u_{xx}\, dx
$$

such that

$$
2 \int uu_x u_{xx}\, dx = -\int u_x^3\, dx. \tag{2.50}
$$

Similarly

$$
\int \frac{1}{2}u^2 u_x\, dx = \int u^2 u_x\, dx
$$

yields

$$
\int u^2 u_x\, dx = 0. \tag{2.51}
$$

Using these two identities the second term on the right-hand side of (2.49) can be expressed as

$$
\begin{aligned}
&\int_0^\infty \int \left(\frac{1}{2}u^2 u_x + \frac{1}{2}u^2 u_{xxx} - u u_x u_{xx}\right) dx \, \zeta \, dt \\
&\overset{(2.51)}{=} \int_0^\infty \int \left(\frac{1}{2}u^2 u_{xxx} - u u_x u_{xx}\right) dx \, \zeta \, dt \\
&= \int_0^\infty \int \left(-\frac{1}{2}\partial_x u^2 u_{xx} - u u_x u_{xx}\right) dx \, \zeta \, dt \\
&= -\int_0^\infty \int 2 u u_x u_{xx} \, dx \, \zeta \, dt \\
&\overset{(2.50)}{=} \int_0^\infty \int u_x^3 \, dx \, \zeta \, dt.
\end{aligned}
\tag{2.52}
$$

Combining Equations (2.48), (2.49) and (2.52) we get

$$
\begin{aligned}
&\int_0^\infty \int \left(\frac{1}{12}u^4 + \frac{1}{2}u^2 g + u_x^3\right) dx \, \zeta \, dt \\
&\overset{(2.52)}{=} \int_0^\infty \int \left(\frac{1}{12}u^4 + \frac{1}{2}u^2 g\right) dx \, \zeta \, dt \\
&\quad + \int_0^\infty \int \left(\frac{1}{2}u^2 u_x + \frac{1}{2}u^2 u_{xxx} - u u_x u_{xx}\right) dx \, \zeta \, dt \\
&\overset{(2.49)}{=} \int_0^\infty \int V \cdot \begin{pmatrix} h_t \\ h_x \end{pmatrix} dx \, \zeta \, dt \\
&\overset{(2.48)}{=} -\int_0^\infty \int h\left(u_x^2 - u_{xx}^2\right) dx \, \zeta \, dt - \int_0^\infty \frac{1}{2}u^2 h \, dx \, \zeta_t \, dt,
\end{aligned}
\tag{2.53}
$$

which, with $g = \frac{1}{L}\int \frac{1}{2}u^2 \, dx$, proves part 1.

In view of part 2 we calculate

$$
\begin{aligned}
-\int \left(u_x^2 - u_{xx}^2\right) h \, dx &= \int \left(u_{xx}^2 h + u u_{xx} h + u u_x h_x\right) dx \\
&\overset{(2.45)}{=} \int \left(u_{xx}^2 h + u u_{xx} h + u^2 u_x\right) dx
\end{aligned}
$$

$$\overset{(2.51)}{=} \int (u_{xx}^2 h + u u_{xx} h) dx$$
$$= \int \left(\frac{1}{2}u + u_{xx}^2\right)^2 h\, dx - \int \frac{1}{4}u^2 h\, dx. \tag{2.54}$$

Plugging this into (2.53) and setting $g = 0$, one gets

$$\begin{aligned}
&\int_0^\infty \int \left(\frac{1}{12}u^4 + u_x^3\right) dx\, \zeta\, dt \\
&\overset{(2.53)}{=} -\int_0^\infty \int h(u_x^2 - u_{xx}^2)\, dx\, \zeta\, dt - \int_0^\infty \frac{1}{2}u^2 h\, dx\, \zeta_t\, dt \\
&\overset{(2.54)}{=} \int_0^\infty \int \left(\frac{1}{2}u + u_{xx}^2\right)^2 h\, dx\, \zeta\, dt - \int_0^\infty \int \frac{1}{4}u^2 h\, dx\, \zeta\, dt \\
&\quad - \int_0^\infty \frac{1}{2}u^2 h\, dx\, \zeta_t\, dt,
\end{aligned}$$

concluding part 2. ■

Now we show that the solutions have regularity $u \in L^4((s, s+T); L^4(0, L))$ for all $s \geq 0$ and $T > 0$.

▶ **Lemma 2.13 (Uniform Integrability).**
Let $L \gtrsim 1$ and u be a solution of (KS), (PC$_\mathrm{L}$), (IC), then for all $s \geq 0$ and $T > 0$

$$\int_s^{s+T} \int u^4\, dxdt \lesssim L^{\frac{1}{2}} \left(e^{\frac{T}{2}} \int u^2(s)\, dx\right)^{\frac{3}{2}}. \tag{2.55}$$

◀

Proof.
Using the translation invariance in time (see Lemma 2.10) we can assume $s = 0$. Let h be defined as in part 1 of Lemma 2.12. This leaves us the freedom of an additional constant which we chose to be such that

$$\int_0^T \int h\, dxdt = 0. \tag{2.56}$$

Then we get

$$\begin{aligned}\frac{d}{dt}\int h\,dx &= \int h_t\,dx\\ &= -\int\left(\frac{1}{2}u^2 + u_x + u_{xxx} - \frac{1}{L}\int \frac{1}{2}u^2(\bar{x})\,d\bar{x}\right)dx\\ &= -\int\left(\frac{1}{2}u^2 + u_x + u_{xxx}\right)dx + \int \frac{1}{2}u^2(\bar{x})\,d\bar{x}\\ &= -\int\left(\frac{1}{2}u^2 + u_x + u_{xxx} - \frac{1}{2}u^2\right)dx\\ &= -\int (u_x + u_{xxx})dx\\ &= 0\end{aligned}$$

by the periodicity of u and therefore u_{xx}. So $\int h\,dx$ is constant in time and by Equation (2.56) we have

$$\int h(x,t)\,dx = 0$$

for all $0 \le t \le T$. With $h_x = u$ and the same arguments as in (2.39) and (2.40) there exists a $y \in [0, L]$ with $h(y,t) = 0$ so that

$$\begin{aligned}|h(x,t)| &= |h(x,t) - h(y,t)|\\ &= \left|\int_y^x h_x\,dx\right|\\ &\le \int |h_x|\,dx\\ &= \int |u|\,dx\\ &\overset{(A.2)}{\le} \|1\|_{L^2([0,L])}\|u\|_{L^2([0,L])}\\ &= \left(L\int u^2(x,t)\,dx\right)^{\frac{1}{2}},\end{aligned}$$

where the right-hand side is independent of x. Therefore

$$\sup_{x\in[0,L]} |h(x)| \leq \left(L \int u^2(x,t)\, dx\right)^{\frac{1}{2}}$$

and

$$\sup_{\substack{x\in[0,L]\\ t\in[0,T]}} |h(x,t)| \leq \left(L \sup_{t\in[0,T]} \int u^2(x,t)\, dx\right)^{\frac{1}{2}}. \tag{2.57}$$

Let ζ be the mollification of

$$\chi_{(0,T)}(x) = \begin{cases} 1 & \text{for } x \in (0,T) \\ 0 & \text{else} \end{cases}, \tag{2.58}$$

the characteristic function of $(0,T)$. Then $\zeta \in L^2(0,T)$ and therefore converges in L^2 meaning that we can use it in part 1 of Lemma 2.12 and get

$$\begin{aligned} &\frac{1}{12}\int_0^T \int u^4\, dxdt + \frac{1}{L}\int_0^T \left(\int \frac{1}{2}u^2\, dx\right)^2 dt + \int_0^T \int u_x^3\, dxdt \\ &\quad = -\int_0^T \int (u_x^2 - u_{xx}^2)h\, dxdt - \int \frac{1}{2}u^2 h\, dx\Big|_0^T, \end{aligned} \tag{2.59}$$

which with the energy estimates (2.31) - (2.34) and (2.57) proves

$$\begin{aligned} &\int_0^T \int u^4\, dxdt \\ &\quad \lesssim \frac{1}{12}\int_0^T \int u^4\, dxdt \\ &\qquad + \frac{1}{L}\int_0^T \left(\int \frac{1}{2}u^2\, dx\right)^2 dt + \int_0^T \int (u_x^3 + |u_x|^3)\, dxdt \\ &\quad \overset{(2.59)}{=} -\int_0^T \int (u_x^2 - u_{xx}^2)h\, dxdt - \int \frac{1}{2}u^2 h\, dx\Big|_{t=0}^T + \int_0^T \int |u_x|^3\, dxdt \end{aligned}$$

$$
\begin{aligned}
&\overset{(2.57)}{\leq} \int_0^T \int |u_x|^3 \, dxdt + \left(L \sup_{t\in[0,T]} \int u^2(t)\, dx \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_0^T \int u_x^2 + u_{xx}^2 \, dxdt + \sup_{t\in[0,T]} \int \frac{1}{2} u^2(t)\, dx \right) \\
&\overset{(2.31)-(2.34)}{\lesssim} \left(1 + L^{\frac{1}{2}}\right) \left(e^{\frac{T}{2}} \int u_0^2 \, dx \right)^{\frac{3}{2}} \\
&\lesssim L^{\frac{1}{2}} \left(e^{\frac{T}{2}} \int u_0^2 \, dx \right)^{\frac{3}{2}},
\end{aligned}
$$

where the last estimate holds because $L \gtrsim 1$ by assumption. ■

▶ **Lemma 2.14 (Initial Layer).**
Let $L \gtrsim 1$ and u be a solution of the initial boundary value problem (KS), (PC_L), (IC), then there exists a constant $c > 0$ such that

$$
\int_0^L u^2(t)\, dx \leq cL^3 \left(1 + \frac{1}{t^2} \right) \tag{2.60}
$$

for all $t > 0$. ◀

Proof.
Let

$$
0 < T \leq 1 \tag{2.61}
$$

and introduce

$$
g(s) := \int_s^{s+T} \int u^2 \, dxdt \tag{2.62}
$$

so that

$$
g'(s) = \int u^2(s+T)\, dx - \int u^2(s)\, dx. \tag{2.63}
$$

For $t \in (s, s+T)$ we get $s+T \geq t$ and therefore by the energy estimate (2.31)

$$\begin{aligned}
T \int u^2(s+T)\,dx &= \int_s^{s+T} dt \int u^2(s+T)\,dx \\
&= \int_s^{s+T} \int u^2(s+T)\,dxdt \\
&\leq \int_s^{s+T} \sup_{\tau \in (t, t+s+T-t)} \int u^2(\tau)\,dxdt \\
&\overset{(2.31)}{\lesssim} \int_s^{s+T} e^{\frac{s+T-t}{2}} \int u^2(t)\,dxdt \\
&\leq e^{\frac{T}{2}} \int_s^{s+T} \int u^2(t)\,dxdt \\
&\lesssim \int_s^{s+T} \int u^2(t)\,dxdt \\
&= g(s).
\end{aligned} \tag{2.64}$$

Using Hölder inequality (see Proposition A.2 in the Appendix) and the uniform integrability (see Lemma 2.13) one gets

$$\begin{aligned}
g(s) &= \int_s^{s+T} \int u^2\,dxdt \\
&\overset{(A.2)}{\leq} \left(\int_s^{s+T} dt\right)^{\frac{1}{2}} \left(\int_s^{s+T} \|u^2(\cdot,t)\|^2_{L^1([0,L])} dt\right)^{\frac{1}{2}} \\
&= T^{\frac{1}{2}} \left(\int_s^{s+T} \|u^2(\cdot,t)\|^2_{L^1([0,L])}\,dt\right)^{\frac{1}{2}} \\
&\overset{(A.2)}{\leq} T^{\frac{1}{2}} \left(\int_s^{s+T} \|1\|^2_{L^2([0,L])} \|u^2(\cdot,t)\|^2_{L^2([0,L])}\,dt\right)^{\frac{1}{2}} \\
&= L^{\frac{1}{2}} T^{\frac{1}{2}} \left(\int_s^{s+T} \int u^4(t)\,dxdt\right)^{\frac{1}{2}} \\
&\overset{(2.55)}{\lesssim} L^{\frac{3}{4}} T^{\frac{1}{2}} \left(e^{\frac{T}{2}} \int u^2(s)\,dx\right)^{\frac{3}{4}}
\end{aligned}$$

$$\overset{(2.61)}{\lesssim} L^{\frac{3}{4}} T^{\frac{1}{2}} \left(\int u^2(s)\, dx \right)^{\frac{3}{4}}, \tag{2.65}$$

which together with (2.63) and (2.64) yields

$$\begin{aligned} \int u^2(s)\, dx &= \int u^2(s)\, dx - \int u^2(s+T)\, dx + \int u^2(s+T)\, dx \\ &\overset{(2.63)}{=} -g'(s) + \int u^2(s+T)\, dx \\ &\overset{(2.64)}{\leq} -g'(s) + \frac{c}{T} g(s). \end{aligned} \tag{2.66}$$

Combining (2.65) and (2.66) we arrive at the differential inequality

$$g(s) \lesssim L^{\frac{3}{4}} T^{\frac{1}{2}} \left(-g'(s) + \frac{c}{T} g(s) \right)^{\frac{3}{4}}. \tag{2.67}$$

Defining

$$\varphi(s) = e^{-\frac{cs}{T}} g(s) \tag{2.68}$$

such that by

$$\varphi'(s) = -\frac{c}{T} e^{-\frac{cs}{T}} g(s) + e^{\frac{-cs}{T}} g'(s) \tag{2.69}$$

we can rewrite (2.67) as

$$\begin{aligned} e^{\frac{cs}{T}} \varphi(s) &\overset{(2.68)}{=} g(s) \\ &\overset{(2.67)}{\lesssim} L^{\frac{3}{4}} T^{\frac{1}{2}} \left(-g'(s) + \frac{c}{T} g(s) \right)^{\frac{3}{4}} \\ &= L^{\frac{3}{4}} T^{\frac{1}{2}} \left(-e^{\frac{cs}{T}} \left(-\frac{c}{T} e^{-\frac{cs}{T}} g(s) + e^{-\frac{cs}{T}} g'(s) \right) \right)^{\frac{3}{4}} \\ &\overset{(2.69)}{=} L^{\frac{3}{4}} T^{\frac{1}{2}} \left(-e^{\frac{cs}{T}} \varphi'(s) \right)^{\frac{3}{4}} \end{aligned} \tag{2.70}$$

and get

$$\left(\frac{3}{\varphi^{\frac{1}{3}}(s)}\right)' = -\frac{\varphi'(s)}{\varphi^{\frac{4}{3}}(s)} \overset{(2.70)}{\gtrsim} T^{-\frac{2}{3}}L^{-1}e^{\frac{cs}{3T}}. \tag{2.71}$$

Since g and therefore φ is by definition non-negative this yields

$$\begin{aligned}\varphi^{-\frac{1}{3}}(s) &\geq \varphi^{-\frac{1}{3}}(s) - \varphi^{-\frac{1}{3}}(s_0)\\ &= \frac{1}{3}\int_{s_0}^{s}\left(3\varphi^{-\frac{1}{3}}(\tau)\right)' d\tau\\ &\overset{(2.71)}{\gtrsim} \frac{1}{3}\int_{s_0}^{s} T^{-\frac{2}{3}}L^{-1}e^{\frac{c\tau}{3T}}\, d\tau\\ &= \frac{1}{3}T^{-\frac{2}{3}}L^{-1}\left(\frac{3T}{c}e^{\frac{c\tau}{3T}}\right)\Bigg|_{s_0}^{s}\\ &\gtrsim T^{\frac{1}{3}}L^{-1}\left(e^{\frac{cs}{3T}} - e^{\frac{cs_0}{3T}}\right).\end{aligned} \tag{2.72}$$

Setting $s_0 = 0$ one obtains

$$\varphi(s) \overset{(2.72)}{\lesssim} L^3T^{-1}\frac{1}{\left(e^{\frac{cs}{3T}} - 1\right)^3} \tag{2.73}$$

and therefore

$$\int u^2(s+T)\, dx \overset{(2.64)}{\lesssim} \frac{1}{T}g(s) \overset{(2.68)}{=} \frac{1}{T}e^{\frac{cs}{T}}\varphi(s) \overset{(2.73)}{\lesssim} \frac{L^3}{T^2}\frac{e^{\frac{cs}{T}}}{\left(e^{\frac{cs}{3T}} - 1\right)^3} \tag{2.74}$$

for all $0 < T \leq 1$ and $s \geq 0$. Now we distinguish between small and large times t.

- For $t \leq 1$ choosing $s = T = \frac{t}{2}$ yields

$$\int u^2(t)\, dx = \int u^2(s+T)\, dx \overset{(2.74)}{\lesssim} \frac{4L^3}{t^2}\frac{e^c}{\left(e^{\frac{c}{3}} - 1\right)^3} \lesssim \frac{L^3}{t^2}. \tag{2.75}$$

- For $t > 1$ choosing $T = \frac{1}{2}$ and $s = t - \frac{1}{2}$ yields $e^{-\frac{c(2t-1)}{3}} < e^{-\frac{c}{3}} < 1$, so that

$$\begin{aligned} \int u^2(t)\,dx &= \int u^2(s+T)\,dx \\ &\overset{(2.74)}{\lesssim} 4L^3 \frac{e^{c(2t-1)}}{\left(e^{\frac{c(2t-1)}{3}} - 1\right)^3} \\ &\lesssim L^3 \frac{1}{\left(1 - e^{-\frac{c(2t-1)}{3}}\right)^3} \\ &\lesssim L^3. \end{aligned} \tag{2.76}$$

Combining Estimates (2.75) and (2.76) implies

$$\int u^2(t)\,dx \lesssim L^3\left(1 + \frac{1}{t^2}\right),$$

which concludes the proof. ■

Next one derives Hölder continuity for h as defined in Equation (2.43).

▶ **Lemma 2.15 (Hölder Continuity of h).**
Let u solve the initial boundary value problem (KS), (PC_L), (IC), h be given by (2.43) as in part 2 of Lemma 2.12 and M be defined by

$$M(T,s) := e^{\frac{T}{2}} \int u^2(s)\,dx. \tag{2.77}$$

Then

$$\begin{aligned} |h(x_1,t_1) - h(x_2,t_2)| \lesssim M^{\frac{1}{2}}|x_1 - x_2|^{\frac{1}{2}} + M^{\frac{2}{3}}|t_1 - t_2|^{\frac{1}{3}} \\ + M^{\frac{1}{2}}|t_1 - t_2|^{\frac{1}{4}} + M^{\frac{1}{2}}|t_1 - t_2|^{\frac{1}{8}} \end{aligned} \tag{2.78}$$

holds for all $x_1, x_2 \in \mathbb{R}$, $s \geq 0$, $T > 0$ and $t_1, t_2 \in (s, s+T)$. ◀

Proof.
By translation invariance in time (see Lemma 2.10) we may assume $s = 0$. We first prove the Hölder continuity in space. Let therefore $t_0 \in (0,T)$ and $x_1, x_2 \in [0,L]$,

then by energy estimate (2.31)

$$\sup_{t\in(0,T)}\int u^2(t)\,dx \overset{(2.31)}{\lesssim} e^{\frac{T}{2}}\int u_0^2\,dx = M, \tag{2.79}$$

such that with Hölder inequality (see Proposition A.2 in the Appendix)

$$\begin{aligned}
|h(x_1,t_0)-h(x_2,t_0)| &= \left|\int_{x_1}^{x_2} h_x(x,t_0)\,dx\right| \\
&\overset{(2.43)}{=} \left|\int_{x_1}^{x_2} u(x,t_0)\,dx\right| \\
&\le \|u(t_0)\|_{L^1(x_1,x_2)} \\
&\overset{(A.2)}{\le} \|1\|_{L^2(x_1,x_2)}\|u(t_0)\|_{L^2(x_1,x_2)} \\
&= |x_1-x_2|^{\frac{1}{2}}\left(\int u^2(t_0)\,dx\right)^{\frac{1}{2}} \\
&\le |x_1-x_2|^{\frac{1}{2}}\left(\sup_{t\in(0,T)}\int u^2(t)\,dx\right)^{\frac{1}{2}} \\
&\overset{(2.79)}{\lesssim} |x_1-x_2|^{\frac{1}{2}}M^{\frac{1}{2}}.
\end{aligned} \tag{2.80}$$

For the Hölder continuity in time we take the standard mollifier

$$\varphi(s) := c^{-1}\chi_{(-1,1)}(s)e^{-\frac{1}{1-s^2}}, \quad 1=\int_{\mathbb{R}}\varphi(s)\,ds, \quad \varphi_\delta(s) := \frac{1}{\delta}\varphi\left(\frac{s}{\delta}\right),$$

such that

$$\varphi_\delta(s)^{(k)} = \delta^{-(k+1)}\varphi^{(k)}\left(\frac{s}{\delta}\right) \lesssim \delta^{-(k+1)} \tag{2.81}$$

and get by the Hölder continuity in space (2.80)

$$\begin{aligned}
&\left|\int \varphi_\delta(x-x_0)h(t,x)\,dx - h(t,x_0)\right| \\
&\quad = \left|\int \varphi_\delta(x-x_0)h(t,x) - \varphi_\delta(x-x_0)h(t,x_0)\,dx\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{x_0-\delta}^{x_0+\delta} \varphi_\delta(x-x_0)|h(t,x)-h(t,x_0)|\,dx \\
&\overset{(2.80)}{\lesssim} M^{\frac{1}{2}} \int_{x_0-\delta}^{x_0+\delta} \varphi_\delta(x-x_0)|x-x_0|^{\frac{1}{2}}\,dx \\
&\leq M^{\frac{1}{2}}\delta^{\frac{1}{2}} \int_{x_0-\delta}^{x_0+\delta} \varphi_\delta(x-x_0)\,dx \\
&= M^{\frac{1}{2}}\delta^{\frac{1}{2}}.
\end{aligned} \tag{2.82}
$$

Using the definition of h, this yields

$$
\begin{aligned}
&|h(x_0,t_1)-h(x_0,t_2)| \\
&\quad= \Big|h(x_0,t_1) - \int \varphi_\delta(x-x_0)h(x,t_1)\,dx \\
&\qquad + \int \varphi_\delta(x-x_0)h(x,t_1) - \varphi_\delta(x-x_0)h(x,t_2) \\
&\qquad + \int \varphi_\delta(x-x_0)h(x,t_2)\,dx - h(x_0,t_2)\Big| \\
&\quad\overset{(2.82)}{\lesssim} 2M^{\frac{1}{2}}\delta^{\frac{1}{2}} + \left|\int \varphi_\delta(x-x_0)(h(x,t_1)-h(x,t_2))dx\right| \\
&\quad\lesssim M^{\frac{1}{2}}\delta^{\frac{1}{2}} + \left|\int_{t_2}^{t_1}\int \varphi_\delta(x-x_0)h_t(x,t)\,dxdt\right| \\
&\quad\overset{(2.43)}{=} M^{\frac{1}{2}}\delta^{\frac{1}{2}} + \left|\int_{t_2}^{t_1}\int \varphi_\delta(x-x_0)\left(\frac{1}{2}u^2+u_x+u_{xxx}\right)dxdt\right|.
\end{aligned} \tag{2.83}
$$

We want to calculate the last integral. Observe that for the quadratic term

$$
\begin{aligned}
\left|\int_{t_2}^{t_1}\int \varphi_\delta(x-x_0)u^2\,dxdt\right| &\leq (\sup\varphi_\delta)\left(\sup_{t\in(t_2,t_1)}\int u^2\,dx\right)\left|\int_{t_2}^{t_1} dt\right| \\
&\overset{(2.81)}{\lesssim} |t_2-t_1|\delta^{-1}\sup_{t\in(0,T)}\int u^2\,dx \\
&\overset{(2.79)}{\lesssim} |t_2-t_1|\delta^{-1}M
\end{aligned} \tag{2.84}
$$

and for the derivative terms, since $0 < t_1, t_2 < T$,

$$
\begin{aligned}
&\left|\int_{t_2}^{t_1}\int \varphi_\delta(x-x_0)(u_x+u_{xxx})\,dxdt\right| \\
&\quad= \left|\int_{t_2}^{t_1}\int ((\varphi_\delta)_x(x-x_0)+(\varphi_\delta)_{xxx}(x-x_0))\; u\,dxdt\right| \\
&\quad\overset{\text{(A.2)}}{\leq} \left|\int_{t_2}^{t_1} \|(\varphi_\delta)_x+(\varphi_\delta)_{xxx}\|_{L^2([-x_0,L-x_0])}\|u\|_{L^2([0,L])}\,dt\right| \\
&\quad\leq \left(\int_{-\delta}^{\delta}((\varphi_\delta)_x+(\varphi_\delta)_{xxx})^2 dx\right)^{\frac{1}{2}}\left|\int_{t_2}^{t_1}\left(\int u^2\,dx\right)^{\frac{1}{2}}dt\right| \\
&\quad\overset{\text{(2.81)}}{\lesssim} (\delta^{-2}+\delta^{-4})\left(\int_{-\delta}^{\delta}dx\right)^{\frac{1}{2}}\left(\sup_{t\in(0,T)}\int u^2\,dx\right)^{\frac{1}{2}}\left|\int_{t_2}^{t_1}dt\right| \\
&\quad\overset{\text{(2.79)}}{\lesssim} (\delta^{-2}+\delta^{-4})\;\delta^{\frac{1}{2}}M^{\frac{1}{2}}|t_1-t_2|
\end{aligned}
\tag{2.85}
$$

holds. Plugging Estimates (2.84) and (2.85) into (2.83) we arrive at

$$
|h(x_0,t_1)-h(x_0,t_2)| \lesssim M^{\frac{1}{2}}\delta^{\frac{1}{2}}\left(1+|t_1-t_2|\left(M^{\frac{1}{2}}\delta^{-\frac{3}{2}}+\delta^{-2}+\delta^{-4}\right)\right). \tag{2.86}
$$

We want the bracket on the right-hand side to be constant. This is accomplished by setting

$$
\delta = \max\left\{M^{\frac{1}{3}}|t_1-t_2|^{\frac{2}{3}}, |t_1-t_2|^{\frac{1}{2}}, |t_1-t_2|^{\frac{1}{4}}\right\}, \tag{2.87}
$$

since then

$$
\delta^{-1} \leq \left(\min\left\{M^{\frac{1}{3}}|t_1-t_2|^{\frac{2}{3}}, |t_1-t_2|^{\frac{1}{2}}, |t_1-t_2|^{\frac{1}{4}}\right\}\right)^{-1}
$$

and therefore

$$
\begin{aligned}
M^{\frac{1}{2}}\delta^{-\frac{3}{2}} &\leq M^{\frac{1}{2}}\left(M^{\frac{1}{3}}|t_1-t_2|^{\frac{2}{3}}\right)^{-\frac{3}{2}} = |t_1-t_2|^{-1}, \\
\delta^{-2} &\leq \left(|t_1-t_2|^{\frac{1}{2}}\right)^{-2} = |t_1-t_2|^{-1}, \\
\delta^{-4} &\leq \left(|t_1-t_2|^{\frac{1}{4}}\right)^{-4} = |t_1-t_2|^{-1}.
\end{aligned}
\tag{2.88}
$$

Combining these estimates yields

$$
\begin{aligned}
|h(x_0,t_1)-h(x_0,t_2)| &\overset{(2.86)}{\lesssim} M^{\frac{1}{2}}\delta^{\frac{1}{2}}\left(1+|t_1-t_2|\left(M^{\frac{1}{2}}\delta^{-\frac{3}{2}}+\delta^{-2}+\delta^{-4}\right)\right)\\
&\overset{(2.88)}{\le} 4M^{\frac{1}{2}}\delta^{\frac{1}{2}}\\
&\overset{(2.87)}{\lesssim} M^{\frac{1}{2}}\left(M^{\frac{1}{3}}|t_1-t_2|^{\frac{2}{3}}+|t_1-t_2|^{\frac{1}{2}}+|t_1-t_2|^{\frac{1}{4}}\right)^{\frac{1}{2}}\\
&\le M^{\frac{2}{3}}|t_1-t_2|^{\frac{1}{3}}+M^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{4}}+M^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{8}}. \qquad (2.89)
\end{aligned}
$$

So h is also Hölder continuous in time. The continuity in space and time imply continuity in space-time, since

$$
\begin{aligned}
|h(x_1,t_1)-h(x_2,t_2)| &= |h(x_1,t_1)-h(x_2,t_1)+h(x_2,t_1)-h(x_2,t_2)|\\
&\le |h(x_1,t_1)-h(x_2,t_1)|+|h(x_2,t_1)-h(x_2,t_2)|\\
&\overset{(2.80),(2.89)}{\lesssim} M^{\frac{1}{2}}|x_1-x_2|^{\frac{1}{2}}+M^{\frac{2}{3}}|t_1-t_2|^{\frac{1}{3}}\\
&\qquad +M^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{4}}+M^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{8}}.
\end{aligned}
$$

■

2.2.2 Convergence of Rescaled Solutions of the Kuramoto-Sivashinsky Equation

Because of the properties derived in Section 2.2.1 we can now state that if the solution u is rescaled according to $u(x,t) = L\tilde{u}(Lx,t) = L\tilde{u}(\tilde{x},\tilde{t})$, then $\tilde{u}$ converges to a solution of the Burgers' equation. The explicit convergence is stated in the following theorem.

▶ **Theorem 2.16 (Compactness).**
Let L_ν be a sequence such that $L_\nu \overset{\nu\to\infty}{\longrightarrow} \infty$ and u_ν a solution of (KS), (PC_L), (IC), where $L = L_\nu$. More precisely fix u_0 with initial conditions (1.10) and let $u_\nu(x,t)$ solve

$$
(u_\nu)_t + \left(\frac{1}{2}u_\nu^2\right)_x + (u_\nu)_{xx} + (u_\nu)_{xxxx} = 0, \qquad (KS_\nu)
$$

$$
u_\nu(x+L_\nu,t) = u_\nu(x,t), \qquad (PC_{L_\nu})
$$

$$u_\nu(x,0) = u_{\nu,0}(x) = u_0\left(\frac{L}{L_\nu}x\right). \tag{IC$_\nu$}$$

Under the rescaling

$$\begin{aligned} x &= L_\nu \hat{x}, \\ t &= \hat{t}, \\ u_\nu(x,t) &= L_\nu \hat{u}_\nu(\hat{x},\hat{t}) \end{aligned} \tag{2.90}$$

there exists a subsequence $\hat{u}_{\nu_\mu}$ of $\hat{u}_\nu$ and a 1-periodic function $\hat{u} \in L^4_{\text{loc}}((0,\infty)\times\mathbb{R})$ with zero average such that

$$\hat{u}_{\nu_\mu} \overset{\mu\to\infty}{\longrightarrow} \hat{u}$$

in $L^4_{\text{loc}}((0,\infty)\times\mathbb{R})$. Additionally $\hat{u}$ solves

$$\hat{u}_{\hat{t}} + \left(\frac{1}{2}\hat{u}^2\right)_{\hat{x}} = 0, \tag{BE}$$

$$\left(\frac{1}{2}\hat{u}^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}^3\right)_{\hat{x}} \le \frac{1}{4}\hat{u}^2 \tag{EC$_{\hat{u}^2}$}$$

in $\mathcal{D}^\star((0,\infty)\times\mathbb{R})$. ◀

Proof.
To omit unnecessary indices we will relabel sequences such that subsequences like $\hat{u}_{\nu_\mu} \overset{\mu\to\infty}{\longrightarrow} \hat{u}$ can be written as $\hat{u}_\nu \overset{\nu\to\infty}{\longrightarrow} \hat{u}$. We will also write C without relabeling it for different constants depending only on T but not on ν.

Weak Convergence
Fix $T > 1$ and let $\hat{t} \in \left(\frac{1}{T}, T\right)$, then Lemma 2.14 implies that there exists a constant $C > 0$ such that

$$\int_0^{L_\nu} u_\nu^2(T^{-1})\,dx \le cL_\nu^3(1+T^2) \lesssim CL_\nu^3. \tag{2.91}$$

Lemma 2.13 states that

$$\int_s^{s+T} \int u_\nu^4 \, dxdt \lesssim L_\nu^{\frac{1}{2}} \left(e^{\frac{T}{2}} \int u_\nu^2(s) \, dx \right)^{\frac{3}{2}}$$

and therefore

$$\begin{aligned} \int_{T^{-1}}^{T} \int u_\nu^4 \, dxdt &\lesssim L_\nu^{\frac{1}{2}} \left(e^{\frac{T-T^{-1}}{2}} \int u_\nu^2(T^{-1}) \, dx \right)^{\frac{3}{2}} \\ &\leq C L_\nu^{\frac{1}{2}} \left(\int u_\nu^2(T^{-1}) \, dx \right)^{\frac{3}{2}} \\ &\overset{(2.91)}{\lesssim} C L_\nu^5. \end{aligned} \tag{2.92}$$

The rescaling (2.90) implies

$$\begin{aligned} \|\hat{u}_\nu\|_{L^4((T^{-1},T);L^4([0,1]))} &= \left(\int_{T^{-1}}^{T} \int_0^1 \hat{u}_\nu^4(\hat{x}, \hat{t}) \, d\hat{x}d\hat{t} \right)^{\frac{1}{4}} \\ &= \left(\int_{T^{-1}}^{T} \int_{\hat{x}(0)}^{\hat{x}(L_\nu)} \hat{u}_\nu^4(\hat{x}, \hat{t}) \, d\hat{x}d\hat{t} \right)^{\frac{1}{4}} \\ &= \left(\int_{T^{-1}}^{T} \int_0^{L_\nu} \hat{u}_\nu^4\left(\frac{x}{L_\nu}, t \right) \frac{1}{L_\nu} dxdt \right)^{\frac{1}{4}} \\ &= \left(\int_{T^{-1}}^{T} \frac{1}{L_\nu^5} \int_0^{L_\nu} u_\nu^4(x,t) dxdt \right)^{\frac{1}{4}} \end{aligned}$$

so that by (2.92)

$$\|\hat{u}_\nu\|_{L^4((T^{-1},T);L^4([0,1]))} \lesssim \left(C \frac{L_\nu^5}{L_\nu^5} \right)^{\frac{1}{4}} = C \tag{2.93}$$

and because of the periodicity condition (PC_{L_ν}), $\hat{u}_\nu$ is a bounded sequence in $L^4((T^{-1},T); L^4_{\mathrm{per}}([0,1]))$. Now we can apply the sequential weak compactness theorem (see Proposition A.11 in the Appendix) and find a subsequence $\hat{u}_\nu$ that is weakly convergent to some $\hat{u} \in L^4((T^{-1},T);L^4_{\text{per}}([0,1]))$. So

$$\hat{u}_\nu \rightharpoonup \hat{u} \tag{2.94}$$

in $L^4((T^{-1},T);L^4_{\text{per}}([0,1]))$, meaning that

$$\lim_{\nu\to\infty} \langle \varphi, \hat{u}_\nu \rangle = \langle \varphi, \hat{u} \rangle$$

holds for every $\varphi \in \left(L^4((T^{-1},T);L^4_{\text{per}}([0,1]))\right)^\star$. Since

$$\left\|\hat{u}_\nu^2\right\|^2_{L^2((T^{-1},T);L^2([0,1]))} = \|\hat{u}_\nu\|^4_{L^4((T^{-1},T);L^4([0,1]))} \overset{(2.93)}{\lesssim} C \tag{2.95}$$

and

$$\left\|\hat{u}_\nu^3\right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((T^{-1},T);L^{\frac{4}{3}}([0,1]))} = \|\hat{u}_\nu\|^4_{L^4((T^{-1},T);L^4([0,1]))} \overset{(2.93)}{\lesssim} C,$$

the sequential weak compactness theorem (see Proposition A.11 in the Appendix) can be applied again and there exist $\overline{\hat{u}^2} \in L^2((T^{-1},T);L^2_{\text{per}}([0,1]))$ and $\overline{\hat{u}^3} \in L^{\frac{4}{3}}((T^{-1},T);L^{\frac{4}{3}}_{\text{per}}([0,1]))$ with

$$\hat{u}_\nu^2 \rightharpoonup \overline{\hat{u}^2} \tag{2.96}$$

in $L^2((T^{-1},T);L^2_{\text{per}}([0,1]))$ and

$$\hat{u}_\nu^3 \rightharpoonup \overline{\hat{u}^3} \tag{2.97}$$

in $L^{\frac{4}{3}}((T^{-1},T);L^{\frac{4}{3}}_{\text{per}}([0,1]))$. Similarly we get

$$\left\|\hat{u}_\nu^4\right\|_{L^1((T^{-1},T);L^1([0,1]))} = \|\hat{u}_\nu\|^4_{L^4((T^{-1},T);L^4([0,1]))} \overset{(2.93)}{\lesssim} C. \tag{2.98}$$

Since L^1 is not reflexive, we can not apply the same procedure to $\hat{u}_\nu^4$. But as L^1 functions can be viewed as weighted Lebesgue measures and therefore Radon measures, (2.98) shows that $\hat{u}_\nu^4$ is a bounded sequence in $\text{RM}((T^{-1},T);\text{RM}_{\text{per}}([0,1]))$. So we can apply the sequential weak star compactness theorem (see Proposition A.12 in the Appendix), which now states that there exists

$\overline{\hat{u}^4} \in \mathrm{RM}((T^{-1}, T); \mathrm{RM}_{\mathrm{per}}([0,1]))$ that fulfills

$$\hat{u}_\nu^4 \overset{\star}{\rightharpoonup} \overline{\hat{u}^4} \tag{2.99}$$

in $\mathrm{RM}((T^{-1}, T); \mathrm{RM}_{\mathrm{per}}([0,1])) \cong \left(C_0^0((T^{-1}, T); C^0([0,1]))\right)^\star$, meaning that

$$\lim_{\nu\to\infty} \langle \hat{u}_\nu^4, \psi\rangle = \langle \overline{\hat{u}^4}, \psi\rangle$$

holds for every $\psi \in C_0^0((T^{-1}, T); C^0([0,1]))$. Under the rescaling (2.90), the Kuramoto-Sivashinsky equation (KS_ν) becomes

$$(\hat{u}_\nu)_{\hat{t}} + \left(\frac{1}{2}\hat{u}_\nu^2\right)_{\hat{x}} + \frac{1}{L_\nu^2}(\hat{u}_\nu)_{\hat{x}\hat{x}} + \frac{1}{L_\nu^4}(\hat{u}_\nu)_{\hat{x}\hat{x}\hat{x}\hat{x}} = 0. \tag{2.100}$$

So $\hat{u}_\nu$ solves (2.100) with initial condition

$$\hat{u}_{0,\nu}(\hat{x}) = \frac{1}{L_\nu} u_{\nu,0}(x) = \frac{1}{L_\nu} u_0\left(\frac{L}{L_\nu}x\right) = \frac{1}{L_\nu} u_0(L\hat{x}).$$

By the choice of the rescaling (2.90), $\hat{x}$ is non dimensional. So derivatives of φ do not increase as $\nu \to \infty$. Therefore the Convergences (2.94) and (2.96) imply that the rescaled Kuramoto-Sivashinsky equation (2.100) converges as

$$\begin{aligned}
0 &= \lim_{\nu\to\infty}\left\langle (\hat{u}_\nu)_{\hat{t}} + \left(\frac{1}{2}\hat{u}_\nu^2\right)_{\hat{x}} + \frac{1}{L_\nu^2}(\hat{u}_\nu)_{\hat{x}\hat{x}} + \frac{1}{L_\nu^4}(\hat{u}_\nu)_{\hat{x}\hat{x}\hat{x}\hat{x}}, \varphi\right\rangle \\
&= -\lim_{\nu\to\infty}\langle \hat{u}_\nu, \varphi_{\hat{t}}\rangle - \lim_{\nu\to\infty}\left\langle \frac{1}{2}\hat{u}_\nu^2, \varphi_{\hat{x}}\right\rangle \\
&\quad + \lim_{\nu\to\infty}\frac{1}{L_\nu^2}\langle \hat{u}_\nu, \varphi_{\hat{x}\hat{x}}\rangle + \lim_{\nu\to\infty}\frac{1}{L_\nu^4}\langle \hat{u}_\nu, \varphi_{\hat{x}\hat{x}\hat{x}\hat{x}}\rangle \\
&\overset{(2.94),(2.96)}{=} -\langle \hat{u}, \varphi_{\hat{t}}\rangle - \left\langle \frac{1}{2}\overline{\hat{u}^2}, \varphi_{\hat{x}}\right\rangle \\
&= \left\langle \hat{u}_{\hat{t}} + \left(\frac{1}{2}\overline{\hat{u}^2}\right)_{\hat{x}}, \varphi\right\rangle
\end{aligned}$$

for every $\varphi \in C_c^\infty((T^{-1},T);C^\infty([0,1]))$. We write this as

$$\hat{u}_{\hat{t}} + \left(\frac{1}{2}\overline{\hat{u}^2}\right)_{\hat{x}} = 0 \tag{2.101}$$

in $\mathcal{D}^\star((T^{-1},T);\mathcal{D}^\star_{\text{per}}([0,1]))$. By the Kuramoto-Sivashinsky equation (KS$_\nu$) we have

$$\begin{aligned}
&\left(\frac{1}{2}u_\nu^2\right)_t + \left(\frac{1}{3}u_\nu^3\right)_x - 2\big((u_{\nu x})^2\big)_{xx} + \left(\frac{1}{2}u_\nu^2\right)_{xxxx}\\
&\quad= u_\nu u_{\nu t} + u_\nu^2 u_{\nu x} - 4\big(u_{\nu xx}^2 + u_{\nu x}u_{\nu xxx}\big)\\
&\qquad + 3u_{\nu xx}^2 + 4u_{\nu x}u_{\nu xxx} + u_\nu u_{\nu xxxx}\\
&\quad\overset{(\text{KS}_\nu)}{=} -u_\nu^2 u_{\nu x} - u_\nu u_{\nu xx} - u_\nu u_{\nu xxxx} + u_\nu^2 u_{\nu x} - 4\big(u_{\nu xx}^2 + u_{\nu x}u_{\nu xxx}\big)\\
&\qquad + 3u_{\nu xx}^2 + 4u_{\nu x}u_{\nu xxx} + u_\nu u_{\nu xxxx}\\
&\quad= -u_\nu u_{\nu xx} - u_{\nu xx}^2\\
&\quad= \frac{1}{4}u_\nu^2 - \left(\frac{1}{2}u_\nu + u_{\nu xx}\right)^2.
\end{aligned} \tag{2.102}$$

Consider the right-hand side of (2.102) as a measure

$$\mu_\nu := \frac{1}{4}u_\nu^2 - \left(\frac{1}{2}u_\nu + u_{\nu xx}\right)^2 \leq \frac{1}{4}u_\nu^2 \tag{2.103}$$

and define the rescaled version so that it matches the rescaling of u in (2.90), i.e.

$$\hat{\mu}_\nu(\hat{x},\hat{t}) := \frac{1}{L_\nu^2}\mu_\nu(x,t) \overset{(2.103)}{\leq} \frac{1}{4L_\nu^2}u_\nu^2(x,t) \overset{(2.90)}{=} \frac{1}{4}\hat{u}_\nu^2(\hat{x},\hat{t}). \tag{2.104}$$

These definitions, the energy estimates (2.31), (2.33) and Inequality (2.91) yield

$$\begin{aligned}
\int_{T^{-1}}^{T}\int_0^1 \big|\hat{\mu}_\nu(\hat{x},\hat{t})\big| d\hat{x}d\hat{t} &\overset{(2.104)}{=} \int_{T^{-1}}^{T}\int_0^1 \frac{1}{L_\nu^2}\big|\mu_\nu(L_\nu\hat{x},\hat{t})\big|\, d\hat{x}d\hat{t}\\
&= \int_{T^{-1}}^{T}\int_0^{L_\nu} \frac{1}{L_\nu^3}|\mu_\nu(x,t)|\, dxdt\\
&\overset{(2.103)}{\leq} \frac{1}{L_\nu^3}\int_{T^{-1}}^{T}\int_0^{L_\nu} \left|\frac{1}{2}u_\nu^2\right| + |u_\nu u_{\nu xx}| + \left|u_{\nu xx}^2\right| dxdt
\end{aligned}$$

$$\begin{aligned}
&\overset{(A.3)}{\lesssim} \frac{1}{L_\nu^3}\int_{T^{-1}}^{T}\int_0^{L_\nu} u_\nu^2 + u_{\nu xx}^2 \, dxdt \\
&\overset{(2.31),(2.33)}{\lesssim} \frac{1}{L_\nu^3} e^{\frac{T-T^{-1}}{2}} \int u_\nu^2(T^{-1}) \, dx \\
&\overset{(2.91)}{\leq} e^{\frac{T-T^{-1}}{2}} C \\
&\leq C.
\end{aligned}$$

Similarly to before, by the weak star compactness theorem (see Proposition A.12 in the Appendix), there exists $\hat{\mu} \in \mathrm{RM}((T^{-1},T);\mathrm{RM}_{\mathrm{per}}([0,1]))$ such that

$$\hat{\mu}_\nu \overset{\star}{\rightharpoonup} \hat{\mu} \tag{2.105}$$

in $\mathrm{RM}((T^{-1},T);\mathrm{RM}_{\mathrm{per}}([0,1])) \cong \left(C_0^0((T^{-1},T);C^0([0,1])\right)^\star$. The Convergences (2.96) and (2.105) together with Inequality (2.104) imply

$$\hat{\mu} \leq \frac{1}{4}\overline{\hat{u}^2} \tag{2.106}$$

in $\mathrm{RM}((T^{-1},T);\mathrm{RM}_{\mathrm{per}}([0,1]))$, meaning

$$\lim_{\nu\to\infty} \langle \hat{\mu}_\nu, \varphi\rangle \leq \left\langle \frac{1}{4}\overline{\hat{u}^2}, \varphi \right\rangle$$

for every non-negative $\varphi \in C_0^0((T^{-1},T);C^0([0,1]))$. Under the rescaling (2.90) and (2.104), Estimates (2.102) and (2.103) give

$$\begin{aligned}
&L_\nu^2\left(\frac{1}{2}\hat{u}_\nu^2\right)_{\hat{t}} + L_\nu^2\left(\frac{1}{3}\hat{u}_\nu^3\right)_{\hat{x}} - 2\left((\hat{u}_{\nu x})^2\right)_{\hat{x}\hat{x}} + \frac{1}{L_\nu^2}\left(\frac{1}{2}\hat{u}_\nu^2\right)_{\hat{x}\hat{x}\hat{x}\hat{x}} \\
&\overset{(2.90)}{=} \left(\frac{1}{2}u_\nu^2\right)_t + \left(\frac{1}{3}u_\nu^3\right)_x - 2\left((u_{\nu x})^2\right)_{xx} + \left(\frac{1}{2}u_\nu^2\right)_{xxxx} \\
&\overset{(2.102)}{=} \frac{1}{4}u_\nu^2 - \left(\frac{1}{2}u_\nu + u_{\nu xx}\right)^2 \\
&\overset{(2.103)}{=} \mu_\nu \\
&\overset{(2.104)}{=} L_\nu^2 \hat{\mu}_\nu,
\end{aligned}$$

where the values of u_ν and μ_ν are to be taken in (x,t) and the ones of $\hat{u}_\nu$ and $\hat{\mu}_\nu$ are to be taken in $(\hat{x},\hat{t})$, which implies

$$\left(\frac{1}{2}\hat{u}_\nu^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}_\nu^3\right)_{\hat{x}} - \frac{2}{L_\nu^2}\left((\hat{u}_{\nu x})^2\right)_{\hat{x}\hat{x}} + \frac{1}{L_\nu^4}\left(\frac{1}{2}\hat{u}_\nu^2\right)_{\hat{x}\hat{x}\hat{x}\hat{x}} = \hat{\mu}_\nu. \tag{2.107}$$

We can estimate the partly unscaled third term on the left-hand side by

$$\begin{aligned}
\int_{T^{-1}}^{T}\int (\hat{u}_{\nu x})^2\, d\hat{x}d\hat{t} &\overset{(2.90)}{=} \frac{1}{L_\nu^3}\int_{T^{-1}}^{T}\int (u_{\nu x})^2\, dxdt \\
&\overset{(2.32)}{\lesssim} \frac{1}{L_\nu^3} e^{\frac{T-T^{-1}}{2}} \int u_\nu^2(T^{-1})\, dx \\
&\overset{(2.91)}{\lesssim} e^{\frac{T-T^{-1}}{2}} C \\
&\leq C.
\end{aligned} \tag{2.108}$$

So that in the limit the Convergences (2.96), (2.97) and (2.105) yield

$$\begin{aligned}
\langle \hat{\mu}, \varphi\rangle &\overset{(2.105)}{=} \lim_{\nu\to\infty} \langle \hat{\mu}_\nu, \varphi\rangle \\
&\overset{(2.107)}{=} \lim_{\nu\to\infty} \Bigg\langle \left(\frac{1}{2}\hat{u}_\nu^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}_\nu^3\right)_{\hat{x}} - \frac{2}{L_\nu^2}\left((\hat{u}_{\nu x})^2\right)_{\hat{x}\hat{x}} \\
&\qquad + \frac{1}{L_\nu^4}\left(\frac{1}{2}\hat{u}_\nu^2\right)_{\hat{x}\hat{x}\hat{x}\hat{x}}, \varphi \Bigg\rangle \\
&= -\lim_{\nu\to\infty}\left\langle \frac{1}{2}\hat{u}_\nu^2, \varphi_{\hat{t}}\right\rangle - \lim_{\nu\to\infty}\left\langle \frac{1}{3}\hat{u}_\nu^3, \varphi_{\hat{x}}\right\rangle - \lim_{\nu\to\infty}\frac{2}{L_\nu^2}\left\langle (\hat{u}_{\nu x})^2, \varphi_{\hat{x}\hat{x}}\right\rangle \\
&\qquad + \lim_{\nu\to\infty}\frac{1}{2L_\nu^4}\left\langle \hat{u}_\nu^2, \varphi_{\hat{x}\hat{x}\hat{x}\hat{x}}\right\rangle \\
&\overset{(2.96),(2.108)}{=} -\lim_{\nu\to\infty}\left\langle \frac{1}{2}\hat{u}_\nu^2, \varphi_{\hat{t}}\right\rangle - \lim_{\nu\to\infty}\left\langle \frac{1}{3}\hat{u}_\nu^3, \varphi_{\hat{x}}\right\rangle \\
&\overset{(2.96),(2.97)}{=} -\left\langle \frac{1}{2}\overline{\hat{u}^2}, \varphi_{\hat{t}}\right\rangle - \left\langle \frac{1}{3}\overline{\hat{u}^3}, \varphi_{\hat{x}}\right\rangle \\
&= \left\langle \left(\frac{1}{2}\overline{\hat{u}^2}\right)_{\hat{t}} + \left(\frac{1}{3}\overline{\hat{u}^3}\right)_{\hat{x}}, \varphi\right\rangle
\end{aligned} \tag{2.109}$$

for every $\varphi \in C_c^\infty((T^{-1}, T); C^\infty([0,1]))$ and by Estimate (2.106)

$$\left\langle \left(\frac{1}{2}\overline{\hat{u}^2}\right)_{\hat{t}} + \left(\frac{1}{3}\overline{\hat{u}^3}\right)_{\hat{x}}, \varphi \right\rangle \leq \left\langle \frac{1}{4}\overline{\hat{u}^2}, \varphi \right\rangle$$

for every non-negative $\varphi \in C_c^\infty((T^{-1}, T); C^\infty([0,1]))$, i.e.

$$\left(\frac{1}{2}\overline{\hat{u}^2}\right)_{\hat{t}} + \left(\frac{1}{3}\overline{\hat{u}^3}\right)_{\hat{x}} = \hat{\mu} \leq \frac{1}{4}\overline{\hat{u}^2} \tag{2.110}$$

in $\mathcal{D}^\star((T^{-1}, T); \mathcal{D}^\star_{\text{per}}([0,1]))$.

Strong Convergence

We will now improve the weak convergence (2.94) into the strong convergence $\hat{u}_\nu \to \hat{u}$ in $L^4((T^{-1}, T); L^4_{\text{per}}([0,1]))$. For this purpose let h_ν be defined as in part 2 of Lemma 2.12, i.e.

$$\begin{pmatrix} (h_\nu)_t \\ (h_\nu)_x \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{2}u_\nu^2 + (u_\nu)_x + (u_\nu)_{xxx}\right) \\ u_\nu \end{pmatrix}, \tag{2.111}$$

normalized similarly to (2.56) by

$$\int_{T^{-1}}^{T} \int h_\nu \, dxdt = 0$$

and rescaled according to

$$\hat{h}_\nu(\hat{x}, \hat{t}) = L_\nu^{-2} h_\nu(x, t). \tag{2.112}$$

For M as in Lemma 2.15 and the time domain (T^{-1}, T) we get

$$M = M(T - T^{-1}, T^{-1}) \overset{(2.77)}{=} e^{\frac{T-T^{-1}}{2}} \int u_\nu^2(T^{-1}) \, dx \overset{(2.91)}{\lesssim} CL_\nu^3 \tag{2.113}$$

so that the Hölder continuity of h_ν (see Lemma 2.15) yields Hölder continuity of $\hat{h}_\nu$, since

$$\begin{aligned} &|\hat{h}_\nu(\hat{x}_1, \hat{t}_1) - \hat{h}_\nu(\hat{x}_2, \hat{t}_2)| \\ &\overset{(2.111)}{=} L_\nu^{-2} |h_\nu(x_1, t_1) - h_\nu(x_2, t_2)| \end{aligned}$$

$$
\begin{aligned}
&\overset{(2.78)}{\lesssim} L_\nu^{-2}\Big(M^{\frac{1}{2}}|x_1-x_2|^{\frac{1}{2}}+M^{\frac{2}{3}}|t_1-t_2|^{\frac{1}{3}}\\
&\qquad\qquad +M^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{4}}+M^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{8}}\Big)\\
&\overset{(2.113)}{\lesssim} CL_\nu^{-2}\Big(L_\nu^{\frac{3}{2}}|x_1-x_2|^{\frac{1}{2}}+L_\nu^{2}|t_1-t_2|^{\frac{1}{3}}\\
&\qquad\qquad +L_\nu^{\frac{3}{2}}|t_1-t_2|^{\frac{1}{4}}+L_\nu^{\frac{3}{2}}|t_1-t_2|^{\frac{1}{8}}\Big)\\
&\overset{(2.90)}{=} C\Big(|\hat{x}_1-\hat{x}_2|^{\frac{1}{2}}+|\hat{t}_1-\hat{t}_2|^{\frac{1}{3}}+L_\nu^{-\frac{1}{2}}|\hat{t}_1-\hat{t}_2|^{\frac{1}{4}}+L_\nu^{-\frac{1}{2}}|\hat{t}_1-\hat{t}_2|^{\frac{1}{8}}\Big)
\end{aligned}
$$

for $\hat{t}_1,\hat{t}_2\in(T^{-1},T)$. Then by Arzelà-Ascoli (see Proposition A.8 in the Appendix) there exists a subsequence $\hat{h}_\nu$ and $\hat{h}\in C^0([T^{-1},T];C^0_{\text{per}}([0,1]))$ such that

$$\hat{h}_\nu\to\hat{h} \tag{2.114}$$

in $C^0([T^{-1},T];C^0_{\text{per}}([0,1]))$. The rescaled version of Equation (2.111) is given by

$$
\begin{aligned}
\begin{pmatrix}(\hat{h}_\nu)_{\hat{t}}\\(\hat{h}_\nu)_{\hat{x}}\end{pmatrix}
&\overset{(2.112)}{=}\begin{pmatrix}L_\nu^{-2}(h_\nu)_t\\L_\nu^{-1}(h_\nu)_x\end{pmatrix}\\
&\overset{(2.111)}{=}\begin{pmatrix}-L_\nu^{-2}\left(\frac{1}{2}u_\nu^2+(u_\nu)_x+(u_\nu)_{xxx}\right)\\L_\nu^{-1}u_\nu\end{pmatrix}\\
&\overset{(2.90)}{=}\begin{pmatrix}-L_\nu^{-2}\left(\frac{1}{2}L_\nu^2\hat{u}_\nu^2+(\hat{u}_\nu)_{\hat{x}}+L_\nu^{-2}(\hat{u}_\nu)_{\hat{x}\hat{x}\hat{x}}\right)\\\hat{u}_\nu\end{pmatrix}\\
&=\begin{pmatrix}-\left(\frac{1}{2}\hat{u}_\nu^2+L_\nu^{-2}(\hat{u}_\nu)_{\hat{x}}+L_\nu^{-4}(\hat{u}_\nu)_{\hat{x}\hat{x}\hat{x}}\right)\\\hat{u}_\nu\end{pmatrix}.
\end{aligned}
\tag{2.115}
$$

The Convergences (2.94) and (2.96) imply

$$
\begin{aligned}
-\langle\hat{h},\varphi_{\hat{t}}\rangle &\overset{(2.114)}{=} -\lim_{\nu\to\infty}\langle\hat{h}_\nu,\varphi_{\hat{t}}\rangle\\
&= \lim_{\nu\to\infty}\langle(\hat{h}_\nu)_{\hat{t}},\varphi\rangle\\
&\overset{(2.115)}{=} \lim_{\nu\to\infty}\left\langle-\left(\frac{1}{2}\hat{u}_\nu^2+L_\nu^{-2}(\hat{u}_\nu)_{\hat{x}}+L_\nu^{-4}(\hat{u}_\nu)_{\hat{x}\hat{x}\hat{x}}\right),\varphi\right\rangle
\end{aligned}
$$

$$= -\frac{1}{2}\lim_{\nu\to\infty}\langle \hat{u}_\nu^2, \varphi\rangle + \lim_{\nu\to\infty}\frac{1}{L_\nu^2}\langle \hat{u}_\nu, \varphi_{\hat{x}}\rangle + \lim_{\nu\to\infty}\frac{1}{L_\nu^4}\langle \hat{u}_\nu, \varphi_{\hat{x}\hat{x}\hat{x}}\rangle$$
$$\overset{(2.94),(2.96)}{=} \left\langle -\frac{1}{2}\overline{\hat{u}^2}, \varphi\right\rangle$$

and

$$\begin{aligned}
-\langle \hat{h}, \varphi_{\hat{x}}\rangle &\overset{(2.114)}{=} -\lim_{\nu\to\infty}\langle \hat{h}_\nu, \varphi_{\hat{x}}\rangle \\
&= \lim_{\nu\to\infty}\langle (\hat{h}_\nu)_{\hat{x}}, \varphi\rangle \\
&\overset{(2.115)}{=} \lim_{\nu\to\infty}\langle \hat{u}_\nu, \varphi\rangle \\
&\overset{(2.94)}{=} \langle \hat{u}, \varphi\rangle
\end{aligned}$$

for every $\varphi \in C_c^\infty((T^{-1},T);C^\infty([0,1]))$. By almost everywhere uniqueness of weak derivatives we get

$$\begin{aligned}
\hat{h}_{\hat{x}} &= \hat{u}, \\
\hat{h}_{\hat{t}} &= -\frac{1}{2}\overline{\hat{u}^2}.
\end{aligned} \tag{2.116}$$

Applying part 2 of Lemma 2.12 yields

$$\begin{aligned}
&\frac{1}{12}\int_{T^{-1}}^{T}\int \hat{u}_\nu^4 \, d\hat{x}\, \zeta \, d\hat{t} + L_\nu^{-5}\int_{T^{-1}}^{T}\int ((u_\nu)_x)^3 \, dx\, \zeta\, dt \\
&\quad = L_\nu^{-5}\frac{1}{12}\int_{T^{-1}}^{T}\int u_\nu^4 \, dx\, \zeta\, dt + L_\nu^{-5}\int_{T^{-1}}^{T}\int ((u_\nu)_x)^3 \, dx\, \zeta\, dt \\
&\quad \overset{(2.44)}{=} L_\nu^{-5}\int_{T^{-1}}^{T}\int \left(\left(\frac{1}{2}u_\nu + (u_\nu)_{xx}\right)^2 - \frac{1}{4}u_\nu^2\right) h_\nu \, dx\, \zeta\, dt \\
&\qquad - L_\nu^{-5}\int_{T^{-1}}^{T}\int \frac{1}{2}u_\nu^2 h_\nu \, dx\, \zeta_t\, dt \\
&\quad \overset{(2.103)}{=} -L_\nu^{-5}\int_{T^{-1}}^{T}\int h_\nu \zeta \, d\mu_\nu(x,t) - L_\nu^{-5}\int_{T^{-1}}^{T}\int \frac{1}{2}u_\nu^2 h_\nu \, dx\, \zeta_t\, dt \\
&\quad \overset{(2.104),(2.112)}{=} -\int_{T^{-1}}^{T}\int \hat{h}_\nu \zeta \, d\hat{\mu}_\nu(\hat{x},\hat{t}) - \int_{T^{-1}}^{T}\int \frac{1}{2}\hat{u}_\nu^2 \hat{h}_\nu \, d\hat{x}\, \zeta_{\hat{t}}\, d\hat{t}
\end{aligned} \tag{2.117}$$

for every $\zeta \in C_c^\infty(T^{-1}, T)$. The unscaled second term on the left-hand side vanishes in the limit, since by the energy estimate (2.34)

$$\begin{aligned} L_\nu^{-5} \int_{T^{-1}}^{T} \int |(u_\nu)_x|^3 \, dx \, \zeta \, dt &\leq \left(\sup_{t \in [T^{-1}, T]} \zeta(t) \right) L_\nu^{-5} \int_{T^{-1}}^{T} \int |(u_\nu)_x|^3 \, dx dt \\ &\overset{(2.34)}{\lesssim} C L_\nu^{-5} \left(e^{\frac{T - T^{-1}}{2}} \int u^2(T^{-1}) \, dx \right)^{\frac{3}{2}} \\ &\leq C L_\nu^{-5} \left(\int u^2(T^{-1}) \, dx \right)^{\frac{3}{2}} \\ &\overset{(2.91)}{\lesssim} C L_\nu^{-5} L_\nu^{\frac{9}{2}} \\ &\longrightarrow 0. \end{aligned} \tag{2.118}$$

So by the Convergences (2.96), (2.99), (2.105), (2.114) and (2.118) in the limit Equation (2.117) becomes

$$\frac{1}{12} \int_{T^{-1}}^{T} \int \zeta \, d\overline{\hat{u}^4}(\hat{x}, \hat{t}) = -\int_{T^{-1}}^{T} \int \hat{h}\zeta \, d\hat{\mu}(\hat{x}, \hat{t}) - \int_{T^{-1}}^{T} \int \frac{1}{2} \overline{\hat{u}^2} \hat{h} \, d\hat{x} \, \zeta_{\hat{t}} \, d\hat{t}. \tag{2.119}$$

$\hat{u}$ is 1-periodic by (PC_{L_ν}) under the rescaling (2.90) and therefore $\hat{\mu}$ and $\hat{h}$ are also 1-periodic. $C_c^\infty((T^{-1}, T); C^\infty([0, 1]))$ is dense in $C_0^0((T^{-1}, T), C^0([0, 1]))$ and $\mu \in \mathrm{RM}((T^{-1}, T); \mathrm{RM_{per}}([0, 1])) \cong \left(C_0^0((T^{-1}, T); C^0([0, 1])\right)^\star$ such that by (2.109)

$$\left\langle \left(\frac{1}{2} \overline{\hat{u}^2} \right)_{\hat{t}} + \left(\frac{1}{3} \overline{\hat{u}^3} \right)_{\hat{x}}, \varphi \right\rangle = \langle \hat{\mu}, \varphi \rangle \tag{2.120}$$

holds for every $\varphi \in C_0^0((T^{-1}, T), C^0([0, 1]))$ and we can calculate the first term on the right-hand side of (2.119) as

$$-\int_{T^{-1}}^{T} \int \hat{h}\zeta \, d\hat{\mu}(\hat{x}, \hat{t}) \overset{(2.120)}{=} -\int_{T^{-1}}^{T} \int \left(\left(\frac{1}{2} \overline{\hat{u}^2} \right)_{\hat{t}} + \left(\frac{1}{3} \overline{\hat{u}^3} \right)_{\hat{x}} \right) \hat{h} \, d\hat{x} \, \zeta \, d\hat{t}$$

$$
\begin{aligned}
&= \int_{T^{-1}}^{T}\int \frac{1}{2}\overline{\hat u^2}\hat h_{\hat t}\,d\hat x\,\zeta\,d\hat t + \int_{T^{-1}}^{T}\int \frac{1}{2}\overline{\hat u^2}\hat h\,d\hat x\,\zeta_{\hat t}\,d\hat t \\
&\qquad + \int_{T^{-1}}^{T}\int \frac{1}{3}\overline{\hat u^3}\hat h_{\hat x}\,d\hat x\,\zeta\,d\hat t \\
&\overset{(2.116)}{=} -\int_{T^{-1}}^{T}\int \left(\frac{1}{2}\overline{\hat u^2}\right)^2 d\hat x\,\zeta\,d\hat t + \int_{T^{-1}}^{T}\int \frac{1}{2}\overline{\hat u^2}\hat h\,d\hat x\,\zeta_{\hat t}\,d\hat t \\
&\qquad + \int_{T^{-1}}^{T}\int \frac{1}{3}\overline{\hat u^3}\hat u\,d\hat x\,\zeta\,d\hat t. \qquad (2.121)
\end{aligned}
$$

Hence combining (2.119) and (2.121) yields

$$
\frac{1}{12}\int_{T^{-1}}^{T}\int \zeta\,d\overline{\hat u^4}(\hat x,\hat t) = -\int_{T^{-1}}^{T}\int \left(\frac{1}{2}\overline{\hat u^2}\right)^2 d\hat x\,\zeta\,d\hat t + \int_{T^{-1}}^{T}\int \frac{1}{3}\hat u\overline{\hat u^3}\,d\hat x\,\zeta\,d\hat t \qquad (2.122)
$$

for every $\zeta \in C_c^\infty([T^{-1},T])$. For $0 \le \eta \in C_c^\infty([T^{-1},T])$ Equation (2.122) and the Convergences (2.94) - (2.99) imply

$$
\begin{aligned}
0 &\le \lim_{\nu\to\infty}\int_{T^{-1}}^{T}\int (\hat u_\nu - \hat u)^4\,d\hat x\,\eta\,d\hat t \\
&= \lim_{\nu\to\infty}\int_{T^{-1}}^{T}\int \left(\hat u_\nu^4 - 4\hat u_\nu^3\hat u + 6\hat u_\nu^2\hat u^2 - 4\hat u_\nu\hat u^3 + \hat u^4\right) d\hat x\,\eta\,d\hat t \\
&\overset{(2.94)-(2.99)}{=} \int_{T^{-1}}^{T}\int \eta\,d\overline{\hat u^4}(\hat x,\hat t) + \int_{T^{-1}}^{T}\int \left(-4\overline{\hat u^3}\hat u + 6\overline{\hat u^2}\hat u^2 - 3\hat u^4\right) d\hat x\,\eta\,d\hat t \\
&\overset{(2.122)}{=} -3\int_{T^{-1}}^{T}\int \left(\left(\overline{\hat u^2}\right)^2 - 2\overline{\hat u^2}\hat u^2 + \hat u^4\right) d\hat x\,\eta\,d\hat t \\
&= -3\int_{T^{-1}}^{T}\int \left(\overline{\hat u^2} - \hat u^2\right)^2 d\hat x\,\eta\,d\hat t \\
&\le 0
\end{aligned}
$$

such that by the fundamental lemma of calculus of variations

$$
\lim_{\nu\to\infty}\|\hat u_\nu - \hat u\|_{L^4((T^{-1},T);L^4([0,1]))} = 0,
$$

which proves the strong convergence

$$\hat{u}_\nu \longrightarrow \hat{u} \tag{2.123}$$

in $L^4((T^{-1},T);L^4_{\text{per}}([0,1]))$. Using Hölder inequality this also gives strong convergence of $\hat{u}_\nu^2 \longrightarrow \hat{u}^2$ in $L^2((T^{-1},T);L^2([0,1]))$ since

$$\begin{aligned}
&\lim_{\nu\to\infty} \|\hat{u}_\nu^2 - \hat{u}^2\|^2_{L^2((T^{-1},T);L^2([0,1]))} \\
&\quad= \lim_{\nu\to\infty} \|(\hat{u}_\nu - \hat{u})(\hat{u}_\nu + \hat{u})\|^2_{L^2((T^{-1},T);L^2([0,1]))} \\
&\quad= \lim_{\nu\to\infty} \|(\hat{u}_\nu - \hat{u})^2(\hat{u}_\nu + \hat{u})^2\|_{L^1((T^{-1},T);L^1([0,1]))} \\
&\quad\overset{(A.2)}{\leq} \lim_{\nu\to\infty} \left(\|(\hat{u}_\nu - \hat{u})^2\|_{L^2((T^{-1},T);L^2([0,1]))}\|(\hat{u}_\nu + \hat{u})^2\|_{L^2((T^{-1},T);L^2([0,1]))}\right) \\
&\quad= \lim_{\nu\to\infty} \left(\|\hat{u}_\nu - \hat{u}\|_{L^4((T^{-1},T);L^4([0,1]))}\|\hat{u}_\nu + \hat{u}\|_{L^4((T^{-1},T);L^4([0,1]))}\right)^2 \\
&\quad\leq \lim_{\nu\to\infty} \Big(\|\hat{u}_\nu - \hat{u}\|_{L^4((T^{-1},T);L^4([0,1]))} \\
&\qquad\quad \left(\|\hat{u}_\nu\|_{L^4((T^{-1},T);L^4([0,1]))} + \|\hat{u}\|_{L^4((T^{-1},T);L^4([0,1]))}\right)\Big)^2 \\
&\quad\overset{(2.93)}{\leq} 2C^{\frac{1}{2}} \lim_{\nu\to\infty} \|\hat{u}_\nu - \hat{u}\|^2_{L^4((T^{-1},T);L^4([0,1]))} \\
&\quad\overset{(2.123)}{=} 0
\end{aligned}$$

and similar the strong convergence $\hat{u}_\nu^3 \longrightarrow \hat{u}^3$ in $L^{\frac{4}{3}}((T^{-1},T);L^{\frac{4}{3}}([0,1]))$ because

$$\begin{aligned}
&\lim_{\nu\to\infty} \|\hat{u}_\nu^3 - \hat{u}^3\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((T^{-1},T);L^{\frac{4}{3}}([0,1]))} \\
&\quad= \lim_{\nu\to\infty} \|(\hat{u}_\nu - \hat{u})(\hat{u}_\nu^2 + \hat{u}^2 + \hat{u}\hat{u}_\nu)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((T^{-1},T);L^{\frac{4}{3}}([0,1]))} \\
&\quad= \lim_{\nu\to\infty} \|(\hat{u}_\nu - \hat{u})^{\frac{4}{3}}(\hat{u}_\nu^2 + \hat{u}^2 + \hat{u}\hat{u}_\nu)^{\frac{4}{3}}\|_{L^1((T^{-1},T);L^1([0,1]))} \\
&\quad\overset{(A.2)}{\leq} \lim_{\nu\to\infty} \Bigg(\|(\hat{u}_\nu - \hat{u})^{\frac{4}{3}}\|_{L^3((T^{-1},T);L^3([0,1]))} \\
&\qquad\qquad \|(\hat{u}_\nu^2 + \hat{u}^2 + \hat{u}\hat{u}_\nu)^{\frac{4}{3}}\|_{L^{\frac{3}{2}}((T^{-1},T);L^{\frac{3}{2}}([0,1]))}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{\nu\to\infty}\Bigg(\Big(\|\hat{u}_\nu - \hat{u}\|^{\frac{4}{3}}_{L^4((T^{-1},T);L^4([0,1]))}\Big)\\
&\qquad\Big(\|\hat{u}_\nu^2 + \hat{u}^2\|^{\frac{4}{3}}_{L^2((T^{-1},T);L^2([0,1]))} + \|\hat{u}^2\hat{u}_\nu^2\|^{\frac{2}{3}}_{L^1((T^{-1},T);L^1([0,1]))}\Big)\Bigg)\\
&\overset{(A.2)}{\leq} \lim_{\nu\to\infty}\Bigg(\Big(\|\hat{u}_\nu - \hat{u}\|^{\frac{4}{3}}_{L^4((T^{-1},T);L^4([0,1]))}\Big)\\
&\qquad\Big(\|\hat{u}_\nu^2\|^{\frac{4}{3}}_{L^2((T^{-1},T);L^2([0,1]))} + \|\hat{u}^2\|^{\frac{4}{3}}_{L^2((T^{-1},T);L^2([0,1]))}\\
&\qquad+\|\hat{u}^2\|^{\frac{2}{3}}_{L^2((T^{-1},T);L^2([0,1]))}\|u_\nu^2\|^{\frac{2}{3}}_{L^2((T^{-1},T);L^2([0,1]))}\Big)\Bigg)\\
&\overset{(2.95)}{\leq} 3C^{\frac{2}{3}} \lim_{\nu\to\infty}\|\hat{u}_\nu - \hat{u}\|^{\frac{4}{3}}_{L^4((T^{-1},T);L^4([0,1]))}\\
&\overset{(2.123)}{=} 0.
\end{aligned}
$$

These strong convergences imply the weak convergences $\hat{u}_\nu^2 \rightharpoonup \hat{u}^2$ in $L^2((T^{-1},T);L^2([0,1]))$ and $\hat{u}_\nu^3 \rightharpoonup \hat{u}^3$ in $L^{\frac{4}{3}}((T^{-1},T);L^{\frac{4}{3}}([0,1]))$. Weak limits are unique, since, for example in the case of $\hat{u}^2$,

$$
\langle\varphi, \overline{\hat{u}^2} - \hat{u}^2\rangle = \langle\varphi, \overline{\hat{u}^2}\rangle - \langle\varphi, \hat{u}^2\rangle \overset{(2.96)}{=} \lim_{\nu\to\infty}\langle\varphi, \hat{u}_\nu^2\rangle - \lim_{\nu\to\infty}\langle\varphi, \hat{u}_\nu^2\rangle = 0
$$

for all $\varphi \in \left(L^2((T^{-1},T);L^2([0,1]))\right)^\star$, implying $\overline{\hat{u}^2} = \hat{u}^2$ and similarly $\overline{\hat{u}^3} = \hat{u}^3$. This turns (2.101) and (2.110) into

$$
\begin{aligned}
\hat{u}_{\hat{t}} + \left(\frac{1}{2}\hat{u}^2\right)_{\hat{x}} &= 0,\\
\left(\frac{1}{2}\hat{u}^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}^3\right)_{\hat{x}} &\leq \frac{1}{4}\hat{u}^2
\end{aligned}
$$

in $\mathcal{D}^\star((T^{-1},T);\mathcal{D}^\star_{\text{per}}([0,1]))$. ■

2.2.3 Entropy Conditions for Burgers' Equation

In this Section we investigate entropy conditions for Burgers' equation based on the work of De Lellis, Otto and Westdickenberg [DOW04]. In order to gain a more general result $\hat{\mu}$ will here be an arbitrary Radon measure with vanishing

$\mathcal{H}^1$-density instead of $\frac{1}{4}\hat{u}^2$ as in the proof of Theorem 2.16. We will show that even such a measure on the right-hand side of (EC$_{\hat{u}^2}$) is negligible.

First we need to define the following form where $\hat{u}$ can be interpreted as a variable.

▶ **Definition 2.17.**
Let σ be a probability measure on $\mathbb{R}$, $v \in L^1(\mathbb{R}, \sigma)$ and the average of v be given by

$$\langle v \rangle = \int_{\mathbb{R}} v(\hat{u})\, d\sigma(\hat{u}).$$

For $f, \eta \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and $q(\hat{u}) = \int_0^{\hat{u}} f'(\lambda)\eta'(\lambda)\, d\lambda$ define the form B as

$$\begin{aligned} B(f,\eta) &= \left\langle \begin{pmatrix} \eta \\ q \end{pmatrix} \cdot \begin{pmatrix} -f \\ \hat{u} \end{pmatrix} \right\rangle - \left\langle \begin{matrix} \eta \\ q \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} -f \\ \hat{u} \end{matrix} \right\rangle \\ &= \langle \hat{u} q \rangle - \langle \eta f \rangle + \langle f \rangle \langle \eta \rangle - \langle \hat{u} \rangle \langle q \rangle. \end{aligned}$$

◀

▶ **Lemma 2.18.**
The form B defined in Definition 2.17 fulfills the following properties.

1. B is bilinear.
2. If $\hat{u}^2$ is μ-summable then
$$\langle (\hat{u} - \langle \hat{u} \rangle)^4 \rangle \leq 3B(\hat{u}^2, \hat{u}^2). \tag{2.124}$$
3. If μ has compact support and f and η are convex then
$$B(f,\eta) \geq 0. \tag{2.125}$$
4. If μ has compact support and $f'', \eta'' \geq 2c$ then
$$3B(f,\eta) \geq c^2 \langle (\hat{u} - \langle \hat{u} \rangle)^4 \rangle. \tag{2.126}$$

◀

Proof.

1. For $f = g + h$ and $\eta = \nu + \zeta$ we get

$$q' = f'\eta' = g'\nu' + g'\zeta' + h'\nu' + h'\zeta' = q'_{g\nu} + q'_{g\zeta} + q'_{h\nu} +'_{h\zeta}$$

and

$$q = q_{g\nu} + q_{g\zeta} + q_{h\nu} + q_{h\zeta}$$

such that

$$\begin{aligned}
B(g + h, \nu + \eta) &= B(f, \eta) \\
&= \langle \hat{u} q \rangle - \langle \eta f \rangle + \langle f \rangle \langle \eta \rangle - \langle \hat{u} \rangle \langle q \rangle \\
&= \langle \hat{u}(q_{g\nu} + q_{g\zeta} + q_{h\nu} + q_{h\zeta}) \rangle - \langle (\nu + \zeta)(g + h) \rangle \\
&\quad + \langle g + h \rangle \langle \nu + \zeta \rangle - \langle \hat{u} \rangle \langle q_{g\nu} + q_{g\zeta} + q_{h\nu} + q_{h\zeta} \rangle \\
&= \langle \hat{u} q_{g\nu} \rangle + \langle \hat{u} q_{g\zeta} \rangle + \langle \hat{u} q_{h\nu} \rangle + \langle \hat{u} q_{h\zeta} \rangle \\
&\quad - \langle \nu g \rangle - \langle \zeta g \rangle - \langle \nu h \rangle - \langle \zeta h \rangle \\
&\quad + \langle g \rangle \langle \nu \rangle + \langle g \rangle \langle \zeta \rangle + \langle h \rangle \langle \nu \rangle + \langle h \rangle \langle \zeta \rangle \\
&\quad - \langle \hat{u} \rangle \langle q_{g\nu} \rangle - \langle \hat{u} \rangle \langle q_{g\zeta} \rangle - \langle \hat{u} \rangle \langle q_{h\nu} \rangle - \langle \hat{u} \rangle \langle q_{h\zeta} \rangle \\
&= B(g, \nu) + B(g, \zeta) + B(h, \nu) + B(h, \zeta). \qquad (2.127)
\end{aligned}$$

For $f = \lambda h$ and $\eta = \tau\zeta$ one has

$$q' = f'\eta' = \lambda\tau h'\zeta' = \lambda\tau q'_{h\zeta}$$

and

$$q = \lambda\tau q_{h\zeta}$$

such that

$$\begin{aligned}
B(\lambda h, \tau\zeta) &= B(f, \eta) \\
&= \langle \hat{u} q \rangle - \langle \eta f \rangle + \langle f \rangle \langle \eta \rangle - \langle \hat{u} \rangle \langle q \rangle \\
&= \langle \hat{u} \lambda\tau q_{h\zeta} \rangle - \langle \tau\zeta\lambda h \rangle + \langle \lambda h \rangle \langle \tau\zeta \rangle - \langle \hat{u} \rangle \langle \lambda\tau q_{h\zeta} \rangle \\
&= \lambda\tau \langle \hat{u} q_{h\zeta} \rangle - \tau\lambda \langle \zeta h \rangle + \lambda\tau \langle h \rangle \langle \zeta \rangle - \lambda\tau \langle \hat{u} \rangle \langle q_{h\zeta} \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \lambda\tau\big(\langle \hat{u} q_{h\zeta}\rangle - \langle h\rangle + \langle h\rangle\langle\zeta\rangle - \langle\hat{u}\rangle\langle q_{h\zeta}\rangle\big) \\
&= \lambda\tau B(h,\zeta). \qquad (2.128)
\end{aligned}
$$

2. Since $f(\hat{u}) = \eta(\hat{u}) = \hat{u}^2$ one gets $q(\hat{u}) = \int_0^{\hat{u}} 2\lambda^2 \, d\lambda = \frac{4}{3}\hat{u}^3$ and

$$
\begin{aligned}
B(\hat{u}^2,\hat{u}^2) &= \frac{4}{3}\langle\hat{u}^4\rangle - \langle\hat{u}^4\rangle + \langle\hat{u}^2\rangle^2 - \frac{4}{3}\langle\hat{u}\rangle\langle\hat{u}^3\rangle \\
&= \frac{1}{3}\langle\hat{u}^4\rangle + \langle\hat{u}^2\rangle^2 - \frac{4}{3}\langle\hat{u}\rangle\langle\hat{u}^3\rangle.
\end{aligned}
$$

Jensen inequality (see Proposition A.4 in the Appendix) yields

$$
\begin{aligned}
\langle(\hat{u}-\langle\hat{u}\rangle)^4\rangle &= \langle(\hat{u}^2 - 2\hat{u}\langle\hat{u}\rangle + \langle\hat{u}\rangle^2)^2\rangle \\
&\overset{(A.4)}{\leq} \langle(\hat{u}^2 - 2\hat{u}\langle\hat{u}\rangle + \langle\hat{u}^2\rangle)^2\rangle \\
&= \langle\hat{u}^4\rangle - 2\langle\hat{u}^3\rangle\langle\hat{u}\rangle + \langle\hat{u}^2\rangle^2 - 2\langle\hat{u}^3\rangle\langle\hat{u}\rangle + 4\langle\hat{u}^2\rangle\langle\hat{u}\rangle^2 \\
&\qquad - 2\langle\hat{u}^2\rangle\langle\hat{u}\rangle^2 + \langle\hat{u}^2\rangle^2 - 2\langle\hat{u}^2\rangle\langle\hat{u}\rangle^2 + \langle\hat{u}^2\rangle^2 \\
&= 3\langle\hat{u}^2\rangle^2 - 4\langle\hat{u}^3\rangle\langle\hat{u}\rangle + \langle\hat{u}^4\rangle \\
&= 3B(\hat{u}^2,\hat{u}^2).
\end{aligned}
$$

3. The set of all convex functions $\eta(\hat{u})$ can be spanned by a family of the linear combinations

$$
c_0\hat{u} + \sum_{i=1}^{p} c_i(\hat{u} - k_i)_{\geq 0},
$$

where $\omega_{\geq 0} = \max\{\omega, 0\}$ and $c_i > 0$ for $i = 1,\dots,p$, see [Daf16, p.178 f.]. So it suffices to check the condition for

$$
\eta_k(\hat{u}) = (\hat{u} - k)_{\geq 0}
$$

for arbitrary $k \in \mathbb{R}$ or equivalently for

$$
\eta_k(\hat{u}) = |\hat{u} - k|. \qquad (2.129)
$$

Similar by [Kru70, p. 240] the condition for the shock admissibility

$$\eta(\hat{u})_{\hat{t}} + q(\hat{u})_{\hat{x}} \leq 0$$

to be fulfilled for arbitrary convex η is equivalent to it being fulfilled for all $\eta_k = |\hat{u} - k|$. So for this proof we assume that η has the form (2.129) and check that $q(\hat{u}) = \operatorname{sign}(\hat{u} - k)(f(\hat{u}) - f(k)) + c_0$, where c_0 is chosen to be such that $q(0) = 0$ is the corresponding q with $q' = f'\eta'$ such that $q(\hat{u}) = \int_0^{\hat{u}} f'(\lambda)\eta'(\lambda)\, d\lambda$. For $\varphi \in \mathcal{D}$ partial integration yields

$$\begin{aligned}\int_{-\infty}^{\infty} q\varphi'\, d\hat{u} &= \int_{-\infty}^{\infty} \operatorname{sign}(\hat{u} - k)(f(\hat{u}) - f(k))\varphi'(\hat{u})\, d\hat{u} \\ &= -\int_{-\infty}^{k} (f(\hat{u}) - f(k))\varphi'(\hat{u})\, d\hat{u} \\ &\quad + \int_{k}^{\infty} (f(\hat{u}) - f(k))\varphi'(\hat{u})\, d\hat{u} \\ &= \int_{-\infty}^{k} f'(\hat{u})\varphi(\hat{u})\, d\hat{u} - \int_{k}^{\infty} f'(\hat{u})\varphi(\hat{u})\, d\hat{u} \\ &= -\int \operatorname{sign}(\hat{u} - k) f'(\hat{u})\varphi(\hat{u})\, d\hat{u} \\ &= -\int \eta'(\hat{u}) f'(\hat{u})\varphi(\hat{u})\end{aligned}$$

such that $q' = f'\eta'$. The constant c_0 does not change the bilinear form since

$$\begin{aligned}\langle \hat{u}(c_0 + q)\rangle - \langle \hat{u}\rangle\langle c_0 + q\rangle &= \langle \hat{u}q\rangle + c_0\langle \hat{u}\rangle - \langle \hat{u}\rangle\langle q\rangle - c_0\langle \hat{u}\rangle \\ &= \langle \hat{u}q\rangle - \langle \hat{u}\rangle\langle q\rangle.\end{aligned}$$

So without loss of generality we assume $c_0 = 0$, get

$$f(\hat{u}) - f(k) - f'(k)(\hat{u} - k) \geq 0 \tag{2.130}$$

by convexity of f and calculate

$$
\begin{aligned}
B(f,\eta) &= \langle(\hat{u}-k)\,\mathrm{sign}(\hat{u}-k)(f(\hat{u})-f(k))\rangle - \langle|\hat{u}-k|\,(f(\hat{u})-f(k))\rangle \\
&\qquad + \langle f(\hat{u})-f(k)\rangle\langle|\hat{u}-k|\rangle \\
&\qquad - \langle\hat{u}-k\rangle\langle\mathrm{sign}(\hat{u}-k)(f(\hat{u})-f(k))\rangle \\
&= \langle f(\hat{u})-f(k)\rangle\langle|\hat{u}-k|\rangle - \langle\hat{u}-k\rangle\langle\mathrm{sign}(\hat{u}-k)(f(\hat{u})-f(k))\rangle \\
&= \langle f(\hat{u})-f(k)\rangle\langle|\hat{u}-k|\rangle - \langle\hat{u}-k\rangle\langle\mathrm{sign}(\hat{u}-k)(f(\hat{u})-f(k))\rangle \\
&\qquad - f'(k)\langle\hat{u}-k\rangle\langle|\hat{u}-k|\rangle + f'(k)\langle\hat{u}-k\rangle\langle\mathrm{sign}(\hat{u}-k)\,(\hat{u}-k)\rangle \\
&= \langle f(\hat{u})-f(k)-f'(k)(\hat{u}-k)\rangle\langle|\hat{u}-k|\rangle \\
&\qquad - \langle\hat{u}-k\rangle\langle\mathrm{sign}(\hat{u}-k)(f(\hat{u})-f(k)-f'(k)(\hat{u}-k))\rangle \\
&\geq \langle f(\hat{u})-f(k)-f'(k)(\hat{u}-k)\rangle\langle|\hat{u}-k|\rangle \\
&\qquad - \langle|\hat{u}-k|\rangle\langle|\mathrm{sign}(\hat{u}-k)(f(\hat{u})-f(k)-f'(k)(\hat{u}-k))|\rangle \\
&= \langle f(\hat{u})-f(k)-f'(k)(\hat{u}-k)\rangle\langle|\hat{u}-k|\rangle \\
&\qquad - \langle|\hat{u}-k|\rangle\langle|f(\hat{u})-f(k)-f'(k)(\hat{u}-k)|\rangle \\
&\overset{(2.130)}{=} \langle f(\hat{u})-f(k)-f'(k)(\hat{u}-k)\rangle\langle|\hat{u}-k|\rangle \\
&\qquad - \langle|\hat{u}-k|\rangle\langle f(\hat{u})-f(k)-f'(k)(\hat{u}-k)\rangle \\
&= 0.
\end{aligned}
$$

4. We consider $f_1(\hat{u}) := f(\hat{u}) - c\hat{u}^2$ and $\eta_1(\hat{u}) := \eta(\hat{u}) - c\hat{u}^2$, which are convex by assumption. Hence by Inequality (2.125) one has $B(f_1,\eta_1)$, $B(f_1,\hat{u}^2)$, $B(\hat{u}^2,\eta_1) \geq 0$ and

$$
\begin{aligned}
B(f,\eta) &= B(f_1 + c\hat{u}^2, \eta_1 + c\hat{u}^2) \\
&\overset{(2.127)}{=} B(f_1,\eta_1) + B(f_1, c\hat{u}^2) + B(c\hat{u}^2,\eta_1) + B(c\hat{u}^2, c\hat{u}^2) \\
&\overset{(2.125)}{\geq} B(c\hat{u}^2, c\hat{u}^2) \\
&\overset{(2.128)}{=} c^2 B(\hat{u}^2,\hat{u}^2) \\
&\overset{(2.124)}{\geq} \frac{c^2}{3}\langle(\hat{u}-\langle\hat{u}\rangle)^4\rangle.
\end{aligned}
$$

■

With this we can now show that for $\hat{u}$, which fulfills (BE), (EC$_{\hat{u}^2}$), $\hat{h}$ is a viscosity solution of the corresponding Hamilton-Jacobi equation.

▶ **Theorem 2.19.**
Let $\Omega \subset \mathbb{R}^2$ be open. If $\hat{u} \in L^4_{\text{loc}}(\Omega)$ satisfies

$$\hat{u}_{\hat{t}} + \left(\frac{\hat{u}^2}{2}\right)_{\hat{x}} = 0, \tag{BE}$$

$$\left(\frac{\hat{u}^2}{2}\right)_{\hat{t}} + \left(\frac{\hat{u}^3}{3}\right)_{\hat{x}} \leq \hat{\mu} \tag{EC$_{\hat{\mu}}$}$$

in $\mathcal{D}'(\Omega)$ with a non-negative Radon measure $\hat{\mu}$ that fulfills

$$\lim_{r \to 0} \frac{\hat{\mu}(B_r(\hat{x}, \hat{t}))}{r} = 0 \tag{2.131}$$

for all $(\hat{x}, \hat{t}) \in \Omega$, then $\hat{h}$ with $\hat{h}_{\hat{t}} = -\frac{\hat{u}^2}{2}$ and $\hat{h}_{\hat{x}} = \hat{u}$ is a viscosity solution of

$$\hat{h}_{\hat{t}} + \frac{\hat{h}_{\hat{x}}^2}{2} = 0. \tag{2.132}$$

◀

Proof.

Continuity
While in the proof of Theorem 2.16 we already showed that $\hat{h}$ is continuous, we will prove it again in order to provide a rigorous proof for this more general theorem. Testing

$$\left(\frac{\hat{u}^2}{2}\right)_{\hat{t}} + \left(\frac{\hat{u}^3}{3}\right)_{\hat{x}} \leq \hat{\mu}$$

with a cut-off function $\psi(\hat{x})$, where $0 \leq \psi \leq 1$, supp$\psi \subset [\hat{x}_0, \hat{x}_1]$ for some $\hat{x}_0$, $\hat{x}_1 \in \mathbb{R}$, $\psi = 1$ on $[\bar{x}_0, \bar{x}_1] \subset [\hat{x}_0, \hat{x}_1]$ and $\psi \in C^\infty$ yields

$$
\begin{aligned}
\int_{\hat{t}_0}^{\hat{t}_1} \int_{\mathbb{R}} \left(\frac{\hat{u}^2}{2}\right)_{\hat{t}} \psi(\hat{x})\, d\hat{x} d\hat{t} - \int_{\hat{t}_0}^{\hat{t}_1} \int_{\mathbb{R}} \frac{\hat{u}^3}{3} \psi_{\hat{x}}(\hat{x})\, d\hat{x} d\hat{t} \\
&= \int_{\hat{t}_0}^{\hat{t}_1} \int_{\mathbb{R}} \left(\frac{\hat{u}^2}{2}\right)_{\hat{t}} \psi(\hat{x})\, d\hat{x} d\hat{t} + \int_{\hat{t}_0}^{\hat{t}_1} \int_{\mathbb{R}} \left(\frac{\hat{u}^3}{3}\right)_{\hat{x}} \psi(\hat{x})\, d\hat{x} d\hat{t} \\
&\leq \int_{\hat{t}_0}^{\hat{t}_1} \int_{\mathbb{R}} \psi(\hat{x})\, d\hat{\mu}(\hat{x}, \hat{t}) \\
&\leq \int_{\hat{t}_0}^{\hat{t}_1} \int_{\hat{x}_0}^{\hat{x}_1} d\hat{\mu}(\hat{x}, \hat{t}) \\
&= \hat{\mu}([\hat{x}_0, \hat{x}_1] \times [\hat{t}_0, \hat{t}_1])
\end{aligned}
$$

such that Hölder inequality (see Proposition A.2 in the Appendix) gives

$$
\begin{aligned}
\int_{\hat{t}_0}^{\hat{t}_1} \frac{d}{d\hat{t}} \int_{\mathbb{R}} \frac{\hat{u}^2}{2} \psi(\hat{x})\, d\hat{x} d\hat{t} &\leq \mu([\hat{x}_0, \hat{x}_1] \times [\hat{t}_0, \hat{t}_1]) + \int_{\hat{t}_0}^{\hat{t}_1} \int_{\mathbb{R}} \frac{\hat{u}^3}{3} \psi_{\hat{x}}(\hat{x})\, d\hat{x} d\hat{t} \\
&\leq \hat{\mu}([\hat{x}_0, \hat{x}_1] \times [\hat{t}_0, \hat{t}_1]) + \sup_{\hat{x} \in [\hat{x}_0, \hat{x}_1]} |\psi_{\hat{x}}(\hat{x})| \int_{t_0}^{t_1} \int_{\hat{x}_0}^{\hat{x}_1} \frac{|\hat{u}|^3}{3}\, d\hat{x} d\hat{t} \\
&\overset{(A.2)}{\leq} \hat{\mu}([\hat{x}_0, \hat{x}_1] \times [\hat{t}_0, \hat{t}_1]) + c \|\hat{u}\|^3_{L^4([\hat{x}_0, \hat{x}_1] \times [\hat{t}_0, \hat{t}_1])} \|1\|_{L^4([\hat{x}_0, \hat{x}_1] \times [\hat{t}_0, \hat{t}_1])} \\
&\leq c.
\end{aligned}
$$

So $\int_{\mathbb{R}} \hat{u}^2 \psi d\hat{x}$ is locally bounded in time, implying $\int_{\hat{x}_0}^{\hat{x}_1} \hat{u}^2 d\hat{x}$ is bounded for all $\hat{x}_0, \hat{x}_1 \in \mathbb{R}$ such that

$$
\hat{u} \in L^\infty_{\text{loc}}(\mathbb{R}; L^2_{\text{loc}}(\mathbb{R})) \tag{2.133}
$$

and since $\hat{h}_{\hat{x}} = \hat{u}$

$$
\hat{h} \in L^\infty_{\text{loc}}(\mathbb{R}; W^{1,2}_{\text{loc}}(\mathbb{R})),
$$

which via Sobolev embedding (see Proposition A.7 in the Appendix) yields

$$
\hat{h} \in L^\infty_{\text{loc}}(\mathbb{R}; C^{0,\frac{1}{2}}_{\text{loc}}(\mathbb{R})). \tag{2.134}
$$

Since $\hat{h}_{\hat{t}} = -\frac{1}{2}\hat{u}^2$ we get by (2.133)

$$\hat{h} \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; L^1_{\text{loc}}(\mathbb{R}))$$

such that the Sobolev embedding implies

$$\hat{h} \in C^{0,1}_{\text{loc}}(\mathbb{R}; L^1_{\text{loc}}(\mathbb{R})). \tag{2.135}$$

Let ζ_ϵ be a mollifier in space, i.e. $\zeta \in \mathcal{D}(\mathbb{R})$ with $\zeta \geq 0$, $\int_{\mathbb{R}} \zeta d\hat{x} = 1$ and additionally $\operatorname{supp}\zeta \subset (-1, 1)$, then for $\zeta_\epsilon(\hat{x}) = \frac{1}{\epsilon}\zeta\left(\frac{\hat{x}}{\epsilon}\right)$

$$\begin{aligned}
&|\hat{h}(\hat{x}, \hat{t}_1) - \hat{h}(\hat{x}, \hat{t}_2)| \\
&\quad \leq |\hat{h}(\hat{x}, \hat{t}_1) - (\hat{h} \star \zeta_\epsilon)(\hat{x}, \hat{t}_1)| \\
&\qquad + |(\hat{h} \star \zeta_\epsilon)(\hat{x}, \hat{t}_1) - (\hat{h} \star \zeta_\epsilon)(\hat{x}, \hat{t}_2)| \\
&\qquad + |(\hat{h} \star \zeta_\epsilon)(\hat{x}, \hat{t}_2) - \hat{h}(\hat{x}, \hat{t}_2)| \\
&\quad \leq \left| \int_{-\epsilon}^{\epsilon} \zeta_\epsilon(\hat{x} - \hat{y})\hat{h}(\hat{x}, \hat{t}_1) - \zeta_\epsilon(\hat{x} - \hat{y})h(\hat{y}, \hat{t}_1)\, d\hat{y} \right| \\
&\qquad + \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \zeta\left(\frac{\hat{y}}{\epsilon}\right) \left| \hat{h}(\hat{x} - \hat{y}, \hat{t}_1) - \hat{h}(\hat{x} - \hat{y}, \hat{t}_2) \right| d\hat{y} \\
&\qquad + \left| \int_{-\epsilon}^{\epsilon} \zeta_\epsilon(\hat{x} - \hat{y})\hat{h}(\hat{y}, \hat{t}_2) - \zeta_\epsilon(\hat{x} - \hat{y})\hat{h}(\hat{x}, \hat{t}_2)\, d\hat{y} \right| \\
&\quad \leq \sup_{|\hat{x} - \hat{y}| \leq \epsilon} |\hat{x} - \hat{y}|^{\frac{1}{2}} \frac{|\hat{h}(\hat{x}, \hat{t}_1) - \hat{h}(\hat{y}, \hat{t}_1)|}{|\hat{x} - \hat{y}|^{\frac{1}{2}}} \\
&\qquad + \sup_{\hat{z} \in \mathbb{R}} \zeta(\hat{z}) \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \left| \hat{h}(\hat{x} - \hat{y}, \hat{t}_1) - \hat{h}(\hat{x} - \hat{y}, \hat{t}_2) \right| d\hat{y} \\
&\qquad + \sup_{|\hat{x} - \hat{y}| \leq \epsilon} |\hat{x} - \hat{y}|^{\frac{1}{2}} \frac{|\hat{h}(\hat{x}, \hat{t}_2) - \hat{h}(\hat{y}, \hat{t}_2)|}{|\hat{x} - \hat{y}|^{\frac{1}{2}}},
\end{aligned}$$

where $\star$ is convolution in space. Using (2.134) we get

$$\sup_{|\hat{x} - \hat{y}| \leq \epsilon} \frac{|\hat{h}(\hat{x}, \hat{t}_i) - \hat{h}(\hat{y}, \hat{t}_i)|}{|\hat{x} - \hat{y}|^{\frac{1}{2}}} \leq c$$

and by (2.135)

$$\int_{-\epsilon}^{\epsilon} |\hat{h}(\hat{x}-\hat{y},\hat{t}_1) - \hat{h}(\hat{x}-\hat{y},\hat{t}_2)|\, d\hat{y} \le c|\hat{t}_1 - \hat{t}_2|$$

such that since $\operatorname{supp}\zeta \le c$

$$|\hat{h}(\hat{x},\hat{t}_1) - \hat{h}(\hat{x},\hat{t}_2)| \le c \sup_{|\hat{x}-\hat{y}|\le\epsilon} |\hat{x}-\hat{y}|^{\frac{1}{2}} + \frac{c|\hat{t}_1-\hat{t}_2|}{\epsilon} = c\left(\epsilon^{\frac{1}{2}} + \frac{|\hat{t}_1-\hat{t}_2|}{\epsilon}\right).$$

Setting $\epsilon = |\hat{t}_2 - \hat{t}_1|^{\frac{2}{3}}$ yields

$$|\hat{h}(\hat{x},\hat{t}_1) - \hat{h}(\hat{x},\hat{t}_2)| \le c|\hat{t}_2 - \hat{t}_1|^{\frac{1}{3}} \tag{2.136}$$

locally in space such that

$$\hat{h} \in C^{0,\frac{1}{3}}_{\mathrm{loc}}(\mathbb{R}; L^{\infty}_{\mathrm{loc}}(\mathbb{R})),$$

implying

$$\hat{h} \in L^{\infty}_{\mathrm{loc}}(\Omega).$$

Combining (2.134) and (2.136) yields

$$\begin{aligned} |\hat{h}(\hat{x}_1,\hat{t}_1) - \hat{h}(\hat{x}_2,\hat{t}_2)| &\le |\hat{h}(\hat{x}_1,\hat{t}_1) - \hat{h}(\hat{x}_1,\hat{t}_2)| + |\hat{h}(\hat{x}_1,\hat{t}_2) - \hat{h}(\hat{x}_2,\hat{t}_2)| \\ &\le c\left(|\hat{t}_1-\hat{t}_2|^{\frac{1}{3}} + |\hat{x}_1-\hat{x}_2|^{\frac{1}{2}}\right) \end{aligned}$$

and therefore one gets Hölder continuity

$$\hat{h} \in C^{0,\frac{1}{3}}_{\mathrm{loc}}(\mathbb{R}; C^{0,\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R})) \tag{2.137}$$

of $\hat{h}$.

Viscosity Subsolution

Next we show by mollification that $\hat{h}$ is a viscosity subsolution of

$$\hat{h}_{\hat{t}} + f(\hat{h}_{\hat{x}}) = \hat{h}_{\hat{t}} + \frac{1}{2}\hat{h}_{\hat{x}}^2 = 0.$$

Let ζ_ϵ be a mollifier in space-time, i.e. $0 \le \zeta \in \mathcal{D}(\mathbb{R}^2)$, $\int_{\mathbb{R}^2} \zeta \, d\hat{x}d\hat{t} = 1$ and $\zeta_\epsilon(\hat{x},\hat{t}) = \frac{1}{\epsilon^2}\zeta\left(\frac{\hat{x}}{\epsilon},\frac{\hat{t}}{\epsilon}\right)$. By definition

$$h_t = -\frac{\hat{u}^2}{2} = -\frac{\hat{h}_{\hat{x}}^2}{2}$$

almost everywhere such that by Jensen inequality (see Proposition A.5 in the Appendix)

$$\begin{aligned}
0 &= \left(\hat{h}_{\hat{t}} + \frac{1}{2}\hat{h}_{\hat{x}}^2\right) \star \zeta_\epsilon \\
&= \hat{h}_{\hat{t}} \star \zeta_\epsilon + \frac{1}{2}\hat{h}_{\hat{x}}^2 \star \zeta_\epsilon \\
&= \hat{h}_{\hat{t}} \star \zeta_\epsilon + \frac{1}{2}\int_{\mathbb{R}^2} \hat{h}_{\hat{x}}^2(\hat{x}-\hat{y},\hat{t}-\hat{k})\zeta_\epsilon(\hat{y},\hat{k}) \, d\hat{y}d\hat{k} \\
&\overset{(A.5)}{\ge} \hat{h}_{\hat{t}} \star \zeta_\epsilon + \frac{1}{2}\left(\int_{\mathbb{R}^2} \hat{h}_{\hat{x}}(\hat{x}-\hat{y},\hat{t}-\hat{k})\zeta_\epsilon(\hat{y},\hat{k}) \, d\hat{y}d\hat{k}\right)^2 \\
&= \hat{h}_{\hat{t}} \star \zeta_\epsilon + \frac{1}{2}(\hat{h}_{\hat{x}} \star \zeta_\epsilon)^2 \\
&= (\hat{h} \star \zeta_\epsilon)_{\hat{t}} + \frac{1}{2}((\hat{h} \star \zeta_\epsilon)_{\hat{x}})^2,
\end{aligned}$$

where $\star$ is the convolution in space and time, implying that $\hat{h}_\epsilon = \hat{h} \star \zeta_\epsilon$ is a classical subsolution. Classical subsolutions are viscosity subsolutions since if $\hat{h}_\epsilon - \xi$ has a minimum at $(\hat{x},\hat{t})$, then

$$\begin{aligned}
\xi_{\hat{t}}(\hat{x},\hat{t}) &= (\hat{h}_\epsilon)_{\hat{t}}(\hat{x},\hat{t}), \\
\xi_{\hat{x}}(\hat{x},\hat{t}) &= (\hat{h}_\epsilon)_{\hat{x}}(\hat{x},\hat{t}),
\end{aligned}$$

implying

$$\xi_{\hat{t}}(\hat{x},\hat{t}) + \frac{1}{2}\xi_{\hat{x}}^2(\hat{x},\hat{t}) = (\hat{h}_\epsilon)_{\hat{t}}(\hat{x},\hat{t}) + \frac{1}{2}(\hat{h}_\epsilon)_{\hat{x}}^2(\hat{x},\hat{t}) \le 0.$$

By (2.137) $\hat{h}$ is continuous such that $\hat{h}_\epsilon = \hat{h} \star \zeta_\epsilon$ converges uniformly to $\hat{h}$ as $\epsilon \longrightarrow 0$ on every compact subset. Therefore by [CL83, Theorem I.2] $\hat{h}$ is a viscosity subsolution of

$$\hat{h}_{\hat{t}} + \frac{1}{2}\hat{h}_{\hat{x}}^2 = 0.$$

Viscosity Supersolution

Next we prove that $\hat{h}$ is also a viscosity supersolution of

$$\hat{h}_{\hat{t}} + \frac{1}{2}\hat{h}_{\hat{x}}^2 = 0.$$

We have to prove that if $\hat{h} - \xi$ has a minimum in $(\hat{x}_0, \hat{t}_0)$ for any smooth ξ, then $\xi_{\hat{t}} + \frac{1}{2}\xi_{\hat{x}}^2 \geq 0$ in $(\hat{x}_0, \hat{t}_0)$. For simplicity we assume that the minimum is in $(\hat{x}_0, \hat{t}_0) = (0, 0)$ with $\hat{h}(0, 0) - \xi(0, 0) = 0$. Additionally we define

$$\zeta_\epsilon(\hat{x}, \hat{t}) = \zeta(\hat{x}, \hat{t}) - \epsilon|(\hat{x}, \hat{t})|,$$

$\Omega_{\epsilon,\delta}$ as the connected component of $\left\{(\hat{x}, \hat{t}) \,\middle|\, (\hat{h} - \xi_\epsilon)(\hat{x}, \hat{t}) < \delta\right\}$ that contains $(0, 0)$, $\langle\cdot\rangle_{\epsilon,\delta}$ as the average over $\Omega_{\epsilon,\delta}$, i.e.

$$\langle f\rangle_{\epsilon,\delta} = \frac{1}{|\Omega_{\epsilon,\delta}|}\int_{\Omega_{\epsilon,\delta}} f \, d\hat{x}d\hat{t}$$

and write

$$f(\epsilon, \delta) \lessapprox g(\epsilon, \delta) \tag{2.138}$$

if there exist constants $c_1, c_2 > 0$ such that $f(\epsilon, \delta) < c_1 g(\epsilon, \delta)$ for $\epsilon, \delta < c_2$ similar to $\lesssim$ before.

First observe that since $\hat{h}$ is continuous and $\hat{h} - \xi$ has a minimum in $(0, 0)$

$$\hat{h} - \xi \geq (\hat{h} - \epsilon)(0, 0) = 0$$

in $\Omega_{\epsilon,\delta}$ provided δ is sufficiently small such that

$$\hat{h} - \xi_\epsilon = \hat{h} - \xi + \epsilon|(\hat{t}, \hat{x})| \geq (\hat{h} - \xi)(0, 0) + \epsilon|(\hat{x}, \hat{t})| = \epsilon|(\hat{x}, \hat{t})|,$$

which yields

$$\Omega_{\epsilon,\delta} \subset \left\{ (\hat{x},\hat{t}) \,\middle|\, \epsilon|(\hat{x},\hat{t})| < \delta \right\} = B_{\frac{\delta}{\epsilon}}(0). \tag{2.139}$$

If δ is sufficiently small $\Omega_{\epsilon,\delta}$ is compactly contained in $B_1(0)$ such that

$$\begin{aligned}
\langle (\hat{h}-\xi_\epsilon)_{\hat{t}} \rangle_{\epsilon,\delta} &= \frac{1}{|\Omega_{\epsilon,\delta}|} \int_{\Omega_{\epsilon,\delta}} (\hat{h}-\xi_\epsilon)_{\hat{t}} \; d\hat{x}d\hat{t} \\
&= \frac{1}{|\Omega_{\epsilon,\delta}|} \int_{B_1(0)} (\hat{h}-\xi_\epsilon)_{\hat{t}} \; \chi_{\{\hat{h}-\xi_\epsilon<\delta\}} \; d\hat{x}d\hat{t} \\
&= \frac{1}{|\Omega_{\epsilon,\delta}|} \int_{B_1(0)} (\hat{h}-\xi_\epsilon-\delta)_{\hat{t}} \; \chi_{\{\hat{h}-\xi_\epsilon-\delta<0\}} \; d\hat{x}d\hat{t} \\
&= \frac{1}{|\Omega_{\epsilon,\delta}|} \int_{B_1(0)} (\min(\hat{h}-\xi_\epsilon-\delta,0))_{\hat{t}} \; d\hat{x}d\hat{t},
\end{aligned}$$

where $\chi_A(\hat{x},\hat{t})$ donates the characteristic function of A. $\hat{h}$ is absolutely continuous with respect to $\hat{t}$ since $\hat{h}_{\hat{t}} = -\frac{1}{2}\hat{u}^2 \in L^2_{\text{loc}}$ such that $\min(\hat{h}-\xi-\delta,0)$ is also absolutely continuous and vanishes in a neighbourhood of ∂B_1, so by the fundamental theorem of calculus

$$\langle (\hat{h}-\xi_\epsilon)_{\hat{t}} \rangle_{\epsilon,\delta} = 0.$$

An analogous calculation shows that $\langle (\hat{h}-\xi_\epsilon)_{\hat{x}} \rangle_{\epsilon,\delta} = 0$ since $\hat{h}_x = \hat{u} \in L^4_{\text{loc}}$ and therefore

$$\left\langle \begin{matrix} -\frac{1}{2}\hat{u}^2 \\ \hat{u} \end{matrix} \right\rangle_{\epsilon,\delta} = \left\langle \begin{matrix} \hat{h}_{\hat{t}} \\ \hat{h}_{\hat{x}} \end{matrix} \right\rangle_{\epsilon,\delta} = \left\langle \begin{matrix} (\hat{h}-\xi_\epsilon)_{\hat{t}} \\ (\hat{h}-\xi_\epsilon)_{\hat{x}} \end{matrix} \right\rangle_{\epsilon,\delta} + \left\langle \begin{matrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{matrix} \right\rangle_{\epsilon,\delta} = \left\langle \begin{matrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{matrix} \right\rangle_{\epsilon,\delta}. \tag{2.140}$$

Since ξ is smooth $|(\xi_\epsilon)_{\hat{t}}| \leq |\xi_{\hat{t}}| + \epsilon \lessapprox 1$ such that

$$\langle \hat{u}^2 \rangle_{\epsilon,\delta} \lessapprox \langle |(\xi_\epsilon)_{\hat{t}}| \rangle_{\epsilon,\delta} \lessapprox 1 \tag{2.141}$$

and Jensen inequality (see Proposition A.4 in the Appendix) yields

$$\langle |\hat{u}| \rangle^2_{\epsilon,\delta} \overset{(A.4)}{\leq} \langle \hat{u}^2 \rangle_{\epsilon,\delta} \lessapprox 1,$$

implying $\langle |\hat{u}| \rangle_{\epsilon,\delta} \lessapprox 1$. Similarly $|(\xi_\epsilon)_{\hat{x}}| \leq |\xi_{\hat{x}}| + \epsilon \lessapprox 1$, which after combination with the previous results, yields

$$\begin{aligned}
\left\langle \left| \begin{matrix} (\hat{h} - \xi_\epsilon)_{\hat{t}} \\ (\hat{h} - \xi_\epsilon)_{\hat{x}} \end{matrix} \right| \right\rangle_{\epsilon,\delta} &\leq \left\langle \left| \begin{matrix} \hat{h}_{\hat{t}} \\ \hat{h}_{\hat{x}} \end{matrix} \right| \right\rangle_{\epsilon,\delta} + \left\langle \left| \begin{matrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{matrix} \right| \right\rangle_{\epsilon,\delta} \\
&= \left\langle \left| \begin{matrix} -\frac{1}{2}\hat{u}^2 \\ \hat{u} \end{matrix} \right| \right\rangle_{\epsilon,\delta} + \left\langle \left| \begin{matrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{matrix} \right| \right\rangle_{\epsilon,\delta} \\
&\lessapprox 1.
\end{aligned} \tag{2.142}$$

In return one gets

$$\begin{aligned}
\int_{\Omega_{\epsilon,\delta}} \left| \begin{matrix} (\min(\hat{h} - \xi_\epsilon - \delta, 0))_{\hat{t}} \\ (\min(\hat{h} - \xi_\epsilon - \delta, 0))_{\hat{x}} \end{matrix} \right| d\hat{x}d\hat{t} &= \int_{\Omega_{\epsilon,\delta}} \left| \begin{matrix} (\hat{h} - \xi_\epsilon)_{\hat{t}} \\ (\hat{h} - \xi_\epsilon)_{\hat{x}} \end{matrix} \right| \chi_{\hat{h}-\xi_\epsilon<\delta} \, d\hat{x}d\hat{t} \\
&= \int_{\Omega_{\epsilon,\delta}} \left| \begin{matrix} (\hat{h} - \xi_\epsilon)_{\hat{t}} \\ (\hat{h} - \xi_\epsilon)_{\hat{x}} \end{matrix} \right| d\hat{x}d\hat{t} \\
&= |\Omega_{\epsilon,\delta}| \left\langle \left| \begin{matrix} (\hat{h} - \xi_\epsilon)_{\hat{t}} \\ (\hat{h} - \xi_\epsilon)_{\hat{x}} \end{matrix} \right| \right\rangle_{\epsilon,\delta} \\
&\overset{(2.142)}{\lessapprox} |\Omega_{\epsilon,\delta}|
\end{aligned}$$

and by Sobolev embedding (see Proposition A.6 in the Appendix)

$$\| \min(\hat{h} - \xi_\epsilon - \delta, 0) \|_{L^2(\Omega_{\epsilon,\delta})} \overset{(A.6)}{\leq} \int_{\Omega_{\epsilon,\delta}} \left| \begin{matrix} (\min(\hat{h} - \xi_\epsilon - \delta, 0))_{\hat{t}} \\ (\min(\hat{h} - \xi_\epsilon - \delta, 0))_{\hat{x}} \end{matrix} \right| d\hat{x}d\hat{t} \lessapprox |\Omega_{\epsilon,\delta}|.$$

Using Hölder inequality we get for $I(\delta)$ as defined in the following

$$\begin{aligned}
I(\delta) &= \| \min(\hat{h} - \xi_\epsilon - \delta, 0) \|_{L^1(\Omega_{\epsilon,\delta})} \\
&\overset{(A.2)}{\leq} \| \min(\hat{h} - \xi_\epsilon - \delta, 0) \|_{L^2(\Omega_{\epsilon,\delta})} \|1\|_{L^2(\Omega_{\epsilon,\delta})} \\
&= \| \min(\hat{h} - \xi_\epsilon - \delta, 0) \|_{L^2(\Omega_{\epsilon,\delta})} |\Omega_{\epsilon,\delta}|^{\frac{1}{2}} \\
&\lessapprox |\Omega_{\epsilon,\delta}|^{\frac{3}{2}}
\end{aligned} \tag{2.143}$$

and since $\nabla(f_+) = \chi_{\{f\geq 0\}}\nabla f$ almost everywhere

$$\begin{aligned}
I(\delta) &= \int_{\Omega_{\epsilon,\delta}} (\xi_\epsilon - \hat{h} + \delta)_+ \, d\hat{x}d\hat{t} \\
&= \int_{\Omega_{\epsilon,\delta}} \int_0^\delta \frac{d}{ds}(\xi_\epsilon - \hat{h} + s)_+ \, ds \, d\hat{x}d\hat{t} \\
&= \int_{\Omega_{\epsilon,\delta}} \int_0^\delta \chi_{\{\xi_\epsilon - \hat{h}+s\geq 0\}} \frac{d}{ds}(\xi_\epsilon - \hat{h} + s) \, ds \, d\hat{x}d\hat{t} \\
&= \int_{\Omega_{\epsilon,\delta}} \int_0^\delta \chi_{\Omega_{\epsilon,s}} \, ds \, d\hat{x}d\hat{t} \\
&= \int_0^\delta \int_{\Omega_{\epsilon,\delta}} \chi_{\Omega_{\epsilon,s}} \, d\hat{x}d\hat{t} \, ds \\
&= \int_0^\delta |\Omega_{\epsilon,s}| \, ds. \qquad (2.144)
\end{aligned}$$

Combining Estimates (2.143) and (2.144) yields

$$I(\delta) \lessapprox |\Omega_{\epsilon,\delta}|^{\frac{3}{2}} = \left(\frac{d}{d\delta}I(\delta)\right)^{\frac{3}{2}},$$

which after rewriting as

$$1 \lessapprox \frac{\left(\frac{d}{d\delta}I(\delta)\right)^{\frac{3}{2}}}{I(\delta)}$$

implies

$$1 \lessapprox \left(\frac{\left(\frac{d}{d\delta}I(\delta)\right)^{\frac{3}{2}}}{I(\delta)}\right)^{\frac{2}{3}} = \frac{\frac{d}{d\delta}I(\delta)}{(I(\delta))^{\frac{2}{3}}} = \frac{d}{d\delta}\left((I(\delta))^{\frac{1}{3}}\right).$$

Therefore δ can be estimated by

$$\delta = \int_0^\delta ds \lessapprox (I(\delta))^{\frac{1}{3}}$$

and because $|\Omega_{\epsilon,\delta}|$ is non-decreasing in δ we get

$$\delta^3 \lessapprox I(\delta) = \int_0^\delta |\Omega_{\epsilon,s}|\, ds \leq |\Omega_{\epsilon,\delta}| \int_0^\delta ds = |\Omega_{\epsilon,\delta}|\delta,$$

implying

$$\delta^2 \lessapprox |\Omega_{\epsilon,\delta}|. \tag{2.145}$$

For δ sufficiently small $\hat{h} - \xi_\epsilon \geq 0$, which yields

$$-\min(\hat{h} - \xi_\epsilon - \delta, 0) = -\min(\hat{h} - \xi_\epsilon, \delta) + \delta \leq \delta \tag{2.146}$$

and by definition $\hat{h}_{\hat{t}} = -\frac{1}{2}\hat{u}^2$, $\hat{h}_{\hat{x}} = \hat{u}$ such that

$$\begin{aligned}
&\left\langle \begin{pmatrix} -\frac{1}{2}\hat{u}^2 \\ \hat{u} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} \right\rangle_{\epsilon,\delta} - \left\langle \begin{pmatrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} \right\rangle_{\epsilon,\delta} \\
&\quad = \left\langle \begin{pmatrix} (\hat{h} - \xi_\epsilon)_{\hat{t}} \\ (\hat{h} - \xi_\epsilon)_{\hat{x}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} \right\rangle_{\epsilon,\delta} \\
&\quad = \frac{1}{|\Omega_{\epsilon,\delta}|} \int_{\Omega_{\epsilon,\delta}} \begin{pmatrix} (\hat{h} - \xi_\epsilon)_{\hat{t}} \\ (\hat{h} - \xi_\epsilon)_{\hat{x}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} d\hat{x}d\hat{t} \\
&\quad = \frac{1}{|\Omega_{\epsilon,\delta}|} \int_{\Omega_{\epsilon,\delta}} \begin{pmatrix} (\min(\hat{h} - \xi_\epsilon - \delta, 0))_{\hat{t}} \\ (\min(\hat{h} - \xi_\epsilon - \delta, 0))_{\hat{x}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} d\hat{x}d\hat{t} \\
&\quad = \frac{1}{|\Omega_{\epsilon,\delta}|} \int_{\Omega_{\epsilon,\delta}} (-\min(\hat{h} - \xi_\epsilon - \delta, 0)) \left(\left(\frac{\hat{u}^2}{2}\right)_{\hat{t}} + \left(\frac{\hat{u}^3}{3}\right)_{\hat{x}} \right) d\hat{x}d\hat{t} \\
&\quad \overset{(2.146)}{\leq} \frac{\delta}{|\Omega_{\epsilon,\delta}|} \int_{\Omega_{\epsilon,\delta}} \left(\left(\frac{\hat{u}^2}{2}\right)_{\hat{t}} + \left(\frac{\hat{u}^3}{3}\right)_{\hat{x}} \right) d\hat{x}d\hat{t} \\
&\quad \overset{(\mathrm{EC}_{\hat{\mu}})}{\leq} \frac{\delta}{|\Omega_{\epsilon,\delta}|} \hat{\mu}(\Omega_{\epsilon,\delta})
\end{aligned}$$

$$\begin{aligned}
&\overset{(2.139)}{\leq} \frac{\delta}{|\Omega_{\epsilon,\delta}|}\hat{\mu}\left(B_{\frac{\delta}{\epsilon}}\right) \\
&\overset{(2.145)}{\lessapprox} \frac{\hat{\mu}\left(B_{\frac{\delta}{\epsilon}}\right)}{\delta}.
\end{aligned} \tag{2.147}$$

We use this estimate, Hölder inequality (see Proposition A.2 in the Appendix) and $\sup_{\Omega_{\epsilon,\delta}}|\nabla\xi_\epsilon| \lessapprox 1$, since ξ_ϵ is smooth, to calculate

$$\begin{aligned}
\langle \hat{u}^4\rangle_{\epsilon,\delta} &\lessapprox \left\langle -\frac{\hat{u}^4}{4} + \frac{\hat{u}^4}{3}\right\rangle_{\epsilon,\delta} \\
&= \left\langle \begin{pmatrix} -\frac{1}{2}\hat{u}^2 \\ u \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} \right\rangle \\
&\overset{(2.147)}{\lessapprox} \left\langle \begin{pmatrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} \right\rangle + \frac{\hat{\mu}(B_{\frac{\delta}{\epsilon}})}{\delta} \\
&\lessapprox \sup_{\Omega_{\epsilon,\delta}} \left| \begin{pmatrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{pmatrix} \right| \left(\langle \hat{u}^2\rangle_{\epsilon,\delta} + \langle |\hat{u}|^3\rangle_{\epsilon,\delta}\right) + \frac{1}{\epsilon}\left(\frac{\epsilon}{\delta}\hat{\mu}\left(B_{\frac{\delta}{\epsilon}}\right)\right) \\
&\overset{(2.131)}{\lessapprox} \langle \hat{u}^2\rangle_{\epsilon,\delta} + \langle |\hat{u}|^3\rangle_{\epsilon,\delta} + \frac{1}{\epsilon} \\
&\overset{(A.2)}{\lessapprox} 1 + \left(\langle \hat{u}^2\rangle_{\epsilon,\delta}\right)^{\frac{1}{2}} \left(\langle \hat{u}^4\rangle_{\epsilon,\delta}\right)^{\frac{1}{2}} + \frac{1}{\epsilon}.
\end{aligned}$$

Applying Cauchy inequality (see Proposition A.3 in the Appendix) we get

$$\begin{aligned}
\langle \hat{u}^4\rangle_{\epsilon,\delta} &\lessapprox 1 + \left(\langle \hat{u}^2\rangle_{\epsilon,\delta}\right)^{\frac{1}{2}} \left(\langle \hat{u}^4\rangle_{\epsilon,\delta}\right)^{\frac{1}{2}} + \frac{1}{\epsilon} \\
&\overset{(A.3)}{\lessapprox} 1 + c\langle \hat{u}^2\rangle_{\epsilon,\delta} + \frac{1}{c}\langle \hat{u}^4\rangle_{\epsilon,\delta} + \frac{1}{\epsilon} \\
&\overset{(2.141)}{\lessapprox} 1 + c + \frac{1}{c}\langle \hat{u}^4\rangle_{\epsilon,\delta} + \frac{1}{\epsilon}
\end{aligned} \tag{2.148}$$

for all $c > 0$. Let c_1 be the constant specified in (2.138) for Estimate (2.148) and set $c = 2c_1$ to get

$$\langle \hat{u}^4\rangle_{\epsilon,\delta} \leq c_1\left(1 + 2c_1 + \frac{1}{2c_1}\langle \hat{u}^4\rangle_{\epsilon,\delta} + \frac{1}{\epsilon}\right) = \frac{1}{2}\langle \hat{u}^4\rangle_{\epsilon,\delta} + c_1\left(1 + 2c_1 + \frac{1}{\epsilon}\right).$$

Subtracting $\frac{1}{2}\langle \hat{u}^4 \rangle_{\epsilon,\delta}$ yields

$$\frac{1}{2}\langle \hat{u}^4 \rangle_{\epsilon,\delta} \leq c_1 \left(1 + 2c_1 + \frac{1}{\epsilon}\right) \lessapprox 1 + \frac{1}{\epsilon} \lessapprox \frac{1}{\epsilon}. \tag{2.149}$$

Using Hölder inequality, this implies

$$\begin{aligned}
\langle |\hat{u}|^3 \rangle_{\epsilon,\delta} &\overset{\text{(A.2)}}{\leq} \left(\langle \hat{u}^2 \rangle_{\epsilon,\delta}\right)^{\frac{1}{2}} \left(\langle \hat{u}^4 \rangle_{\epsilon,\delta}\right)^{\frac{1}{2}} \\
&\overset{\text{(2.141)}}{\lessapprox} \left(\langle \hat{u}^4 \rangle_{\epsilon,\delta}\right)^{\frac{1}{2}} \\
&\overset{\text{(2.149)}}{\lessapprox} \epsilon^{-\frac{1}{2}}.
\end{aligned} \tag{2.150}$$

Having estimates for $\hat{u}^2$, $\hat{u}^3$ and using the definition of ξ_ϵ, i.e.

$$\xi_\epsilon(\hat{x}, \hat{t}) = \xi(\hat{x}, \hat{t}) - \epsilon|(\hat{x}, \hat{t})|, \tag{2.151}$$

one gets

$$\begin{aligned}
&\left| \left\langle \begin{pmatrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} \right\rangle_{\epsilon,\delta} - \left\langle \begin{matrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{matrix} \right\rangle_{\epsilon,\delta} \left\langle \begin{matrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{matrix} \right\rangle_{\epsilon,\delta} \right| \\
&\quad \leq \left\langle \left| \begin{matrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{matrix} \right| \right\rangle_{\epsilon,\delta} \sup_{\Omega_{\epsilon,\delta}} \left| \begin{pmatrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{pmatrix} - \left\langle \begin{matrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{matrix} \right\rangle_{\epsilon,\delta} \right| \\
&\quad \lessapprox \left(\left\langle \hat{u}^2 \right\rangle_{\epsilon,\delta} + \left\langle |\hat{u}|^3 \right\rangle_{\epsilon,\delta} \right) \Omega_{\epsilon,\delta} \text{osc} \left| \begin{pmatrix} \xi \\ (\xi_\epsilon)_{\hat{x}} \end{pmatrix} \right| \\
&\quad \overset{\text{(2.141),(2.150)}}{\lessapprox} \left(1 + \epsilon^{-\frac{1}{2}}\right) \operatorname*{osc}_{\Omega_{\epsilon,\delta}} \left| \begin{pmatrix} (\xi_\epsilon)_{\hat{t}} \\ (\xi_\epsilon)_{\hat{x}} \end{pmatrix} \right| \\
&\quad \overset{\text{(2.151)}}{\lessapprox} \left(1 + \epsilon^{-\frac{1}{2}}\right) \left(\epsilon + \operatorname*{osc}_{\Omega_{\epsilon,\delta}} \left| \begin{pmatrix} \xi_{\hat{t}} \\ \xi_{\hat{x}} \end{pmatrix} \right| \right).
\end{aligned} \tag{2.152}$$

Since ξ is smooth $\operatorname*{osc}_{\Omega_{\epsilon,\delta}} |\nabla \xi| \leq c \operatorname{diam}(\Omega_{\epsilon,\delta})$, which because of $\Omega_{\epsilon,\delta} \subset B_{\frac{\delta}{\epsilon}}(0)$ by (2.139) yields $\operatorname*{osc}_{\Omega_{\epsilon,\delta}} |\nabla \xi| \lessapprox \frac{\delta}{\epsilon}$ and therefore

$$
\begin{aligned}
&\left|\left\langle\begin{pmatrix}(\xi_\epsilon)_{\hat t}\\(\xi_\epsilon)_{\hat x}\end{pmatrix}\cdot\begin{pmatrix}\frac12\hat u^2\\\frac13\hat u^3\end{pmatrix}\right\rangle_{\epsilon,\delta}-\left\langle\begin{matrix}(\xi_\epsilon)_{\hat t}\\(\xi_\epsilon)_{\hat x}\end{matrix}\right\rangle_{\epsilon,\delta}\left\langle\begin{matrix}\frac12\hat u^2\\\frac13\hat u^3\end{matrix}\right\rangle_{\epsilon,\delta}\right|\\
&\qquad\overset{(2.152)}{\lessapprox}\left(1+\epsilon^{-\frac12}\right)\left(\epsilon+\operatorname*{osc}_{\Omega_{\epsilon,\delta}}\left|\begin{pmatrix}\xi_{\hat t}\\\xi_{\hat x}\end{pmatrix}\right|\right)\\
&\qquad\lessapprox\left(1+\epsilon^{-\frac12}\right)\left(\epsilon+\frac{\delta}{\epsilon}\right).
\end{aligned}\tag{2.153}
$$

For the form B defined in Definition 2.17 we get

$$
\begin{aligned}
B(f,\eta)&=B\left(\frac12\hat u^2,\frac12\hat u^2\right)\\
&=\left\langle\begin{pmatrix}-\frac12\hat u^2\\\hat u\end{pmatrix}\cdot\begin{pmatrix}\frac12\hat u^2\\\frac13\hat u^3\end{pmatrix}\right\rangle_{\epsilon,\delta}-\left\langle\begin{matrix}-\frac12\hat u^2\\\hat u\end{matrix}\right\rangle_{\epsilon,\delta}\left\langle\begin{matrix}\frac12\hat u^2\\\frac13\hat u^3\end{matrix}\right\rangle_{\epsilon,\delta}\\
&\overset{(2.140)}{=}\left\langle\begin{pmatrix}-\frac12\hat u^2\\\hat u\end{pmatrix}\cdot\begin{pmatrix}\frac12\hat u^2\\\frac13\hat u^3\end{pmatrix}\right\rangle_{\epsilon,\delta}-\left\langle\begin{matrix}(\xi_\epsilon)_{\hat t}\\(\xi_\epsilon)_{\hat x}\end{matrix}\right\rangle_{\epsilon,\delta}\left\langle\begin{matrix}\frac12\hat u^2\\\frac13\hat u^3\end{matrix}\right\rangle_{\epsilon,\delta}\\
&=\left\langle\begin{pmatrix}-\frac12\hat u^2\\\hat u\end{pmatrix}\cdot\begin{pmatrix}\frac12\hat u^2\\\frac13\hat u^3\end{pmatrix}\right\rangle_{\epsilon,\delta}-\left\langle\begin{pmatrix}(\xi_\epsilon)_{\hat t}\\(\xi_\epsilon)_{\hat x}\end{pmatrix}\cdot\begin{pmatrix}\frac12\hat u^2\\\frac13\hat u^3\end{pmatrix}\right\rangle_{\epsilon,\delta}\\
&\qquad+\left\langle\begin{pmatrix}(\xi_\epsilon)_{\hat t}\\(\xi_\epsilon)_{\hat x}\end{pmatrix}\cdot\begin{pmatrix}\frac12\hat u^2\\\frac13\hat u^3\end{pmatrix}\right\rangle_{\epsilon,\delta}-\left\langle\begin{matrix}(\xi_\epsilon)_{\hat t}\\(\xi_\epsilon)_{\hat x}\end{matrix}\right\rangle_{\epsilon,\delta}\left\langle\begin{matrix}\frac12\hat u^2\\\frac13\hat u^3\end{matrix}\right\rangle_{\epsilon,\delta}\\
&\overset{(2.147),(2.153)}{\lessapprox}\frac{\hat\mu\left(B_{\frac{\delta}{\epsilon}}\right)}{\delta}+\frac{\delta}{\epsilon^{\frac32}}+\epsilon^{\frac12}.
\end{aligned}
$$

Therefore by part 4 of Lemma 2.18 we get

$$
\langle(\hat u-\langle\hat u\rangle_{\epsilon,\delta})^4\rangle_{\epsilon,\delta}\le B(f,\eta)\lessapprox\frac{\hat\mu\left(B_{\frac{\delta}{\epsilon}}\right)}{\delta}+\frac{\delta}{\epsilon^{\frac32}}+\epsilon^{\frac12}.\tag{2.154}
$$

Since ξ is smooth and $\Omega_{\epsilon,\delta}\subset B_{\frac{\delta}{\epsilon}}$ we calculate

$$
\left|\xi_{\hat t}(0,0)+\frac12\xi_{\hat x}^2(0,0)-\langle\xi_{\hat t}\rangle_{\epsilon,\delta}-\frac12\left(\langle\xi_{\hat x}\rangle_{\epsilon,\delta}\right)^2\right|\lessapprox\operatorname*{osc}_{\Omega_{\epsilon,\delta}}(\xi_{\hat t})+\operatorname*{osc}_{\Omega_{\epsilon,\delta}}(\xi_{\hat x}^2)\lessapprox\frac{\delta}{\epsilon}.
$$

Using Hölder inequality and $\epsilon^2 \lessapprox \epsilon$ we finally arrive at

$$\begin{aligned}
\left|\xi_{\hat{t}}(0,0) + \frac{1}{2}\xi_{\hat{x}}^2(0,0)\right| &\lessapprox \left|\langle \xi_{\hat{t}}\rangle_{\epsilon,\delta} + \frac{1}{2}\big(\langle \xi_{\hat{x}}\rangle_{\epsilon,\delta}\big)^2\right| + \frac{\delta}{\epsilon} \\
&\lessapprox \left|\langle (\xi_\epsilon)_{\hat{t}}\rangle_{\epsilon,\delta} + \frac{1}{2}\big(\langle (\xi_\epsilon)_{\hat{x}}\rangle_{\epsilon,\delta}\big)^2\right| + \epsilon + \epsilon^2 + \frac{\delta}{\epsilon} \\
&\overset{(2.140)}{\lessapprox} \left|-\frac{1}{2}\langle \hat{u}^2\rangle_{\epsilon,\delta} + \frac{1}{2}\big(\langle \hat{u}\rangle_{\epsilon,\delta}\big)^2\right| + \epsilon + \frac{\delta}{\epsilon} \\
&\lessapprox \frac{1}{2}\left|-\langle \hat{u}^2\rangle_{\epsilon,\delta} + 2(\langle \hat{u}\rangle_{\epsilon,\delta})^2 - \big(\langle \hat{u}\rangle_{\epsilon,\delta}\big)^2\right| + \epsilon + \frac{\delta}{\epsilon} \\
&= \frac{1}{2}\left|\langle \hat{u}^2\rangle_{\epsilon,\delta} - 2(\langle \hat{u}\rangle_{\epsilon,\delta})^2 + \big(\langle \hat{u}\rangle_{\epsilon,\delta}\big)^2\right| + \epsilon + \frac{\delta}{\epsilon} \\
&= \frac{1}{2}\left|\langle (\hat{u} - \langle \hat{u}\rangle_{\epsilon,\delta})^2\rangle_{\epsilon,\delta}\right| + \epsilon + \frac{\delta}{\epsilon} \\
&\overset{(A.2)}{\lessapprox} \frac{1}{2}\left|\langle (\hat{u} - \langle \hat{u}\rangle_{\epsilon,\delta})^4\rangle_{\epsilon,\delta}\right|^{\frac{1}{2}} + \epsilon + \frac{\delta}{\epsilon} \\
&\overset{(2.154)}{\lessapprox} \left(\frac{\hat{\mu}\big(B_{\frac{\delta}{\epsilon}}\big)}{\delta} + \frac{\delta}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}\right)^{\frac{1}{2}} + \epsilon + \frac{\delta}{\epsilon}
\end{aligned}$$

such that for $\delta \longrightarrow 0$

$$\left|\xi_{\hat{t}}(0,0) + \frac{1}{2}\xi_{\hat{x}}^2(0,0)\right| \lessapprox \epsilon^{\frac{1}{4}} + \epsilon$$

by (2.131) and in the limit $\epsilon \longrightarrow 0$

$$\xi_{\hat{t}}(0,0) + \frac{1}{2}\xi_{\hat{x}}^2(0,0) = 0,$$

implying $\hat{h}$ is a viscosity supersolution, which concludes the proof. ■

Now we show that $\hat{h}$ being a viscosity solution of $\hat{h}_{\hat{t}} + \frac{1}{2}\hat{h}_{\hat{x}}^2 = 0$ implies that $\hat{u}$ is an entropy solution of $\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 0$.

▶ **Corollary 2.20.**
The function $\hat{u}$ as defined in Theorem 2.19 is an entropy solution of

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 0.$$

◀

Sketch of Proof.
By definition $\hat{u} \in L^4_{\mathrm{loc}}(\Omega)$ and therefore Hölder inequality (see Proposition A.2 in the Appendix) yields $\hat{u} \in L^1_{\mathrm{loc}}(\Omega)$.

We now show that $\hat{u}$ can only have decreasing jumps which implies the necessary entropy condition. For simplicity we assume that $\hat{u}$ is smooth outside of the jump set curve $\gamma = \left\{(\hat{x}(\hat{t}), \hat{t}) \,\middle|\, \hat{t} \geq 0\right\}$ and that it can only jump across this curve. Let $\hat{u}^+$ be the value of $\hat{u}$ on the right and u^- be the value of $\hat{u}$ on the left side of γ and similar $\hat{h}^\pm$. Therefore we have to show that

$$\hat{u}^+ \leq \hat{u}^- \tag{2.155}$$

on γ. Since $\hat{h}$ is continuous and $\hat{u}$ may only jump across γ, we get that $\hat{h}^\pm$ are classical solutions of $\hat{h}^\pm_{\hat{t}} + \frac{1}{2}\left(\hat{h}^\pm_{\hat{x}}\right)^2 = 0$ outside of γ and $\hat{h}^+(\hat{x}(\hat{t}), \hat{t}) = \hat{h}^-(\hat{x}(\hat{t}), \hat{t})$ for all $\hat{t} \geq 0$ such that

$$\frac{d}{d\hat{t}}\hat{h}^+(\hat{x}(\hat{t}), \hat{t}) = \frac{d}{d\hat{t}}\hat{h}^-(\hat{x}(\hat{t}), \hat{t}),$$

which yields the Rankine-Hugoniot condition

$$\begin{aligned}
0 &= \frac{d}{d\hat{t}}\left(\hat{h}^+(\hat{x}(\hat{t}), \hat{t}) - \hat{h}^-(\hat{x}(\hat{t}), \hat{t})\right) \\
&= \hat{x}'(\hat{t})\left(\hat{h}^+_{\hat{x}}(\hat{x}(\hat{t}), \hat{t}) - \hat{h}^-_{\hat{x}}(\hat{x}(\hat{t}), \hat{t})\right) + \hat{h}^+_{\hat{t}}(\hat{x}(\hat{t}), \hat{t})) - \hat{h}^-_{\hat{t}}(\hat{x}(\hat{t}), \hat{t}) \\
&= \hat{x}'(\hat{t})\left(\hat{h}^+_{\hat{x}}(\hat{x}(\hat{t}), \hat{t}) - \hat{h}^-_{\hat{x}}(\hat{x}(\hat{t}), \hat{t})\right) - \frac{1}{2}\left(\left(\hat{h}^+_{\hat{x}}(\hat{x}(\hat{t}), \hat{t})\right)^2 - \left(\hat{h}^-_{\hat{x}}(\hat{x}(\hat{t}), \hat{t})\right)^2\right).
\end{aligned} \tag{2.156}$$

Fix $\hat{T} > 0$ and $\hat{X} = \hat{x}(\hat{T})$, then $(\hat{X}, \hat{T}) \in \gamma$. We will show

$$\hat{h}^+_{\hat{x}}(\hat{X}, \hat{T}) \leq \hat{h}^-_{\hat{x}}(\hat{X}, \hat{T}) \tag{2.157}$$

by contradiction, so suppose

$$\hat{h}^+_{\hat{x}}(\hat{X},\hat{T}) > \hat{h}^-_{\hat{x}}(\hat{X},\hat{T}). \tag{2.158}$$

For $\rho \in \left(\hat{h}^-_{\hat{x}}(\hat{X},\hat{T}), \hat{h}^+_{\hat{x}}(\hat{X},\hat{T})\right)$ set

$$\xi(\hat{x},\hat{t}) = \hat{h}^-(\hat{x}(\hat{t}),\hat{t}) + \rho(\hat{x} - \hat{x}(\hat{t})) \tag{2.159}$$

such that

$$\begin{aligned} \xi_{\hat{t}}(\hat{x},\hat{t}) &= \hat{x}'(\hat{t})\hat{h}^-_{\lambda}(\lambda,\hat{t})\big|_{\lambda=\hat{x}(\hat{t})} + \hat{h}^-_{\hat{t}}(\hat{x}(\hat{t}),\hat{t}) - \rho x'(\hat{t}), \\ \xi_{\hat{x}}(\hat{x},\hat{t}) &= \rho \end{aligned} \tag{2.160}$$

and

$$\hat{h} - \xi = \hat{h}(\hat{x},\hat{t}) - \hat{h}^-(\hat{x}(\hat{t}),\hat{t}) - \rho(\hat{x} - \hat{x}(\hat{t})).$$

Calculating

$$\begin{aligned} (\hat{h} - \xi)(\hat{x}(\hat{t}),\hat{t}) &= \hat{h}(\hat{x}(\hat{t}),\hat{t}) - \hat{h}^-(\hat{x}(\hat{t}),\hat{t}) - \rho(\hat{x}(\hat{t}) - \hat{x}(\hat{t})) \\ &= \hat{h}^-(\hat{x}(\hat{t}),\hat{t}) - \hat{h}^-(\hat{x}(\hat{t}),\hat{t}) - \rho(\hat{x}(\hat{t}) - \hat{x}(\hat{t})) \\ &= 0 \end{aligned}$$

shows that $\hat{h} - \xi$ is constant in γ and, by

$$(\hat{h} - \xi)^+_{\hat{x}} = \hat{h}^+_{\hat{x}} - \rho > 0, \qquad (\hat{h} - \xi)^-_{\hat{x}} = \hat{h}^-_{\hat{x}} - \rho < 0,$$

increasing when going to the left or right side of the jump set, implying that $\hat{h} - \xi$ has a minimum in $(\hat{X},\hat{T})$. Now calculate

$$\begin{aligned} &\xi_{\hat{t}}(\hat{X},\hat{T}) + \frac{1}{2}\xi^2_{\hat{x}}(\hat{X},\hat{T}) \\ &\overset{(2.160)}{=} \hat{x}'(\hat{T})\hat{h}^-_{\hat{x}}(\hat{X},\hat{T}) + \hat{h}^-_{\hat{t}}(\hat{X},\hat{T}) - \rho x'(\hat{T}) + \frac{1}{2}\rho^2 \\ &= -\hat{x}'(\hat{T})\left(\rho - \hat{h}^-_{\hat{x}}(\hat{X},\hat{T})\right) + \hat{h}^-_{\hat{t}}(\hat{X},\hat{T}) + \frac{1}{2}\rho^2 \\ &= -\hat{x}'(\hat{T})\left(\rho - \hat{h}^-_{\hat{x}}(\hat{X},\hat{T})\right) - \frac{1}{2}\left(\hat{h}^-_{\hat{x}}(\hat{X},\hat{T})\right)^2 + \frac{1}{2}\rho^2, \end{aligned} \tag{2.161}$$

implying that the right-hand side of (2.161) vanishes for $\rho = \hat{h}^-_{\hat{x}}(\hat{X},\hat{T})$ and also for $\rho = \hat{h}^+_{\hat{x}}(\hat{X},\hat{T})$ because of the Rankine-Hugoniot condition (2.156). The right-hand side is also strictly convex in ρ, which since $\hat{h}^-_{\hat{x}}(\hat{X},\hat{T}) < \rho < \hat{h}^+_{\hat{x}}(\hat{X},\hat{T})$, implies

$$\xi_{\hat{t}}(\hat{X},\hat{T}) + \frac{1}{2}\xi^2_{\hat{x}}(\hat{X},\hat{T}) < 0,$$

contradicting the assumption that $\hat{h}$ is a viscosity solution, since $\hat{h} - \xi$ has a minimum in $(\hat{X},\hat{T})$.

Therefore the assumption (2.158) is wrong and the claim (2.157) is proven. Since $\hat{h}^\pm_{\hat{x}} = \hat{u}^\pm$ this shows that $\hat{u}$ can only have decreasing jumps and therefore fulfills (2.155).

Based on [Daf16, Chapter 8.5] we translate (2.155) into the condition

$$-s(\eta(\hat{u}^+) - \eta(\hat{u}^-)) + q(\hat{u}^+) - q(\hat{u}^-) \leq 0, \tag{2.162}$$

where the shock speed s is given by

$$s = \frac{1}{2}\frac{(\hat{u}^+)^2 - (\hat{u}^-)^2}{\hat{u}^+ - \hat{u}^-} = \frac{1}{2}\hat{u}^+ + \frac{1}{2}\hat{u}^-. \tag{2.163}$$

Similar to the proof of part 3 in Lemma 2.18 it is sufficient to check (2.162) for $\eta(\hat{u}) = (\hat{u} - k)_{\geq 0}$, where $\omega_{\geq 0} = \max\{\omega, 0\}$ in order for the condition to hold for every entropy-entropy flux pair. The corresponding q is given by

$$q(\hat{u}) = \frac{1}{2}\chi_{\{\hat{u}-k\geq 0\}}(\hat{u}^2 - k^2),$$

where χ_A is the characteristic function of A. For $k \leq \hat{u}^+ \leq \hat{u}^-$ one gets

$$\begin{aligned}
&-s(\eta(\hat{u}^+) - \eta(\hat{u}^-)) + q(\hat{u}^+) - q(\hat{u}^-) \\
&\quad = -\frac{1}{2}\big(\hat{u}^+ + \hat{u}^-\big)\big((\hat{u}^+ - k)_{\geq 0} - (\hat{u}^- - k)_{\geq 0}\big) \\
&\qquad + \frac{1}{2}\chi_{\{\hat{u}^+-k\geq 0\}}\big((\hat{u}^+)^2 - k^2\big) - \frac{1}{2}\chi_{\{\hat{u}^--k\geq 0\}}\big((\hat{u}^-)^2 - k^2\big) \\
&\quad = -\frac{1}{2}\big(\hat{u}^+ + \hat{u}^-\big)\big(\hat{u}^+ - k - (\hat{u}^- - k)\big) + \frac{1}{2}\big((\hat{u}^+)^2 - k^2\big) - \frac{1}{2}\big((\hat{u}^-)^2 - k^2\big) \\
&\quad = 0,
\end{aligned}$$

for $\hat{u}^+ < k \le \hat{u}^-$

$$\begin{aligned}
&-s(\eta(\hat{u}^+) - \eta(\hat{u}^-)) + q(\hat{u}^+) - q(\hat{u}^-) \\
&\quad = -\frac{1}{2}\left(\hat{u}^+ + \hat{u}^-\right)\left((\hat{u}^+ - k)_{\ge 0} - (\hat{u}^- - k)_{\ge 0}\right) \\
&\qquad + \frac{1}{2}\chi_{\{\hat{u}^+ - k \ge 0\}}\left((\hat{u}^+)^2 - k^2\right) - \frac{1}{2}\chi_{\{\hat{u}^- - k \ge 0\}}\left((\hat{u}^-)^2 - k^2\right) \\
&\quad = \frac{1}{2}\left(\hat{u}^+ + \hat{u}^-\right)(\hat{u}^- - k) - \frac{1}{2}\left((\hat{u}^-)^2 - k^2\right) \\
&\quad = \frac{1}{2}\left(\hat{u}^+ + \hat{u}^-\right)(\hat{u}^- - k) - \frac{1}{2}(\hat{u}^- - k)(\hat{u}^- + k) \\
&\quad = \frac{1}{2}\left(\hat{u}^+ - k\right)(\hat{u}^- - k) \\
&\quad \le 0
\end{aligned}$$

and for $\hat{u}^+ \le \hat{u}^- < k$ trivially $q(\hat{u}^\pm) = 0$ and $\eta(\hat{u}^\pm) = 0$, proving (2.162).

Since $\hat{u}$ is smooth outside of γ the entropy condition is satisfied since

$$\begin{aligned}
\eta(\hat{u})_{\hat{t}} + q(\hat{u})_{\hat{x}} &= \eta'(\hat{u})\hat{u}_{\hat{t}} + q'(\hat{u})\hat{u}_{\hat{x}} \\
&= \eta'(\hat{u})(\hat{u}_{\hat{t}} + f'(\hat{u})\hat{u}_{\hat{x}}) \\
&= \eta'(\hat{u})(\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}}) \\
&= 0
\end{aligned}$$

by the definition of q'. Calculating

$$\begin{aligned}
0 &\ge \eta(\hat{u})_{\hat{t}} + q(\hat{u})_{\hat{x}} \\
&= \eta'(\hat{u})\hat{u}_{\hat{t}} + q'(\hat{u})\hat{u}_{\hat{x}} \\
&= -\eta'(\hat{u})\hat{u}\hat{u}_{\hat{x}} + q'(\hat{u})\hat{u}_{\hat{x}} \\
&= (-\eta'(\hat{u})f'(\hat{u}) + q'(\hat{u}))\hat{u}_{\hat{x}}
\end{aligned}$$

with $f(\hat{u}) = \frac{1}{2}\hat{u}^2$ shows that on the jump set curve γ the condition is given by

$$\begin{aligned}
0 &\ge \left(-\frac{\eta(\hat{u}^+) - \eta(\hat{u}^-)}{\hat{u}^+ - \hat{u}^-}\frac{f(\hat{u}^+) - f(\hat{u}^-)}{\hat{u}^+ - \hat{u}^-} + \frac{q(\hat{u}^+) - q(\hat{u}^-)}{\hat{u}^+ - \hat{u}^-}\right)\left(u^+ - u^-\right) \\
&= -\frac{1}{2}\left(\eta(\hat{u}^+) - \eta(\hat{u}^-)\right)\frac{(\hat{u}^+)^2 - (\hat{u}^-)^2}{\hat{u}^+ - \hat{u}^-} + \left(q(\hat{u}^+) - q(\hat{u}^-)\right)
\end{aligned}$$

$$= -s\big(\eta(\hat{u}^+) - \eta(\hat{u}^-)\big) + \big(q(\hat{u}^+) - q(\hat{u}^-)\big),$$

which is condition (2.162), that has already been proven to be fulfilled. ■

2.2.4 Energy Bound for Burgers' Equation

Theorems 2.16 and 2.19 together with Corollary 2.20 show that the rescaled solution of the Kuramoto-Sivashinsky equation fulfills

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 0,$$
$$\eta(\hat{u})_{\hat{t}} + q(\hat{u})_{\hat{x}} \le 0$$

in $\mathcal{D}^\star((0,\infty)\times\mathbb{R})$ for every entropy-entropy flux pair. Choosing this pair to be

$$\eta(\hat{u}) = \frac{1}{2}\hat{u}^2,$$
$$q(\hat{u}) = \frac{1}{3}\hat{u}^3,$$

we get a bound of type

$$\int \hat{u}^2(\hat{x},\hat{t})\,d\hat{x} \le \mathcal{O}\big(\hat{t}^{-2}\big).$$

▶ **Lemma 2.21 (Energy Bound for Burgers' Equation).**
Let $\hat{u} \in L^4_{\text{loc}}((0,\infty)\times\mathbb{R})$ be a solution of

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} = 0, \tag{BE}$$
$$\left(\frac{1}{2}\hat{u}^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}^3\right)_{\hat{x}} \le 0 \tag{EC$_0$}$$

in $\mathcal{D}^\star((0,\infty)\times\mathbb{R})$ with 1-periodicity condition

$$\hat{u}(\hat{x},\hat{t}) = \hat{u}(\hat{x}+1,\hat{t}) \tag{PC$_1$}$$

for all $\hat{x} \in \mathbb{R}$ and $\hat{t} > 0$ and initial condition

$$\hat{u}(\hat{x},0) = \hat{u}_0(\hat{x}), \tag{IC}$$

where the initial data is chosen to match the 1-periodicity, to be locally square-integrable and have zero mean, i.e.

$$\begin{aligned} \hat{u}_0(\hat{x}) &= \hat{u}_0(\hat{x}+1), \\ \int_0^1 \hat{u}_0(\hat{x})\, d\hat{x} &= 0, \\ \hat{u}_0 &\in L^2(0,1) \end{aligned} \tag{2.164}$$

for all $\hat{x} \in \mathbb{R}$. Then there exists a universal constant $c_0 > 0$ such that

$$\int_0^1 \hat{u}^2(\hat{x},\hat{t})\, d\hat{x} \leq \frac{c_0}{\hat{t}^2}$$

for all $\hat{t} > 0$. ◀

Proof.
Until now $\hat{h}$ is only defined through

$$\begin{pmatrix} \hat{h}_{\hat{t}} \\ \hat{h}_{\hat{x}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\hat{u}^2 \\ \hat{u} \end{pmatrix}. \tag{2.165}$$

leaving us the choice of an additive constant. We define $\hat{h}_0(\hat{x}) = \hat{h}(\hat{x},0)$ and set the constant such that

$$\sup_{\hat{x}\in[0,1]} \hat{h}_0(\hat{x}) = 0. \tag{2.166}$$

Using these definitions and Hölder inequality (see Proposition A.2 in the Appendix), the calculation

$$\begin{aligned} -\left(\int \hat{u}_0^2\, d\hat{x}\right)^{\frac{1}{2}} &\overset{(2.165)}{=} -\|(\hat{h}_0)_{\hat{x}}\|_{L^2([0,1])} \\ &\overset{(A.2)}{\leq} -\int_0^1 |(\hat{h}_0)_{\hat{x}}|\, d\hat{x} \\ &\leq -\left(\sup_{\hat{x}\in[0,1]} \hat{h}_0(\hat{x}) - \inf_{\hat{x}\in[0,1]} \hat{h}_0(\hat{x})\right) \\ &\overset{(2.166)}{=} \inf_{\hat{x}\in[0,1]} \hat{h}_0(\hat{x}) \\ &\leq \hat{h}_0(\hat{x}) \end{aligned}$$

$$\leq \sup_{\hat{x}\in[0,1]} \hat{h}_0(\hat{x}) \overset{(2.166)}{=} 0 \tag{2.167}$$

implies

$$\hat{h}(\hat{x},\hat{t}) = \int_0^{\hat{t}} \hat{h}_{\hat{t}}(\hat{x},\hat{\tau})\, d\hat{\tau} + \hat{h}_0(\hat{x}) \overset{(2.165)}{=} -\int_0^{\hat{t}} \frac{1}{2}\hat{u}^2(\hat{x},\hat{\tau})\, d\hat{\tau} + \hat{h}_0(\hat{x}) \overset{(2.167)}{\leq} 0, \tag{2.168}$$

so that not only h is not merely non-positive for $t = 0$ but rather for all t. Since $\hat{u}$ is 1-periodic so is $\hat{h}$, so that by partial integration and the entropy condition (EC_0) we get

$$\begin{aligned}
\frac{1}{12}\int_0^{\hat{T}}\int \hat{u}^4\, d\hat{x}d\hat{t} &= \int_0^{\hat{T}}\int \begin{pmatrix} -\frac{1}{2}\hat{u}^2 \\ \hat{u} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} d\hat{x}d\hat{t} \\
&\overset{(2.165)}{=} \int_0^{\hat{T}}\int \begin{pmatrix} \hat{h}_{\hat{t}} \\ \hat{h}_{\hat{x}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\hat{u}^2 \\ \frac{1}{3}\hat{u}^3 \end{pmatrix} d\hat{x}d\hat{t} \\
&= -\int_0^{\hat{T}}\int \hat{h}\left(\left(\frac{1}{2}\hat{u}^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}^3\right)_{\hat{x}}\right) d\hat{x}d\hat{t} \\
&\quad + \frac{1}{2}\int \hat{h}(\hat{x},\hat{T})\hat{u}^2(\hat{x},\hat{T})\, d\hat{x} - \frac{1}{2}\int \hat{h}_0\hat{u}_0^2\, d\hat{x} \\
&\overset{(2.168),(EC_0)}{\leq} -\frac{1}{2}\int \hat{h}_0\hat{u}_0^2\, d\hat{x} \\
&\overset{(2.167)}{\leq} \frac{1}{2}\left(\int \hat{u}_0^2\, d\hat{x}\right)^{\frac{3}{2}}
\end{aligned}$$

for arbitrary $\hat{T} > 0$. Similarly to Lemma 2.10, the Burgers' equation is also translation invariant. Using this and the fact that $\hat{T}$ was arbitrary we get

$$\int_{\hat{s}}^{\infty}\int \hat{u}^4\, d\hat{x}d\hat{t} \leq 6\left(\int \hat{u}^2(\hat{x},\hat{s})\, d\hat{x}\right)^{\frac{3}{2}} \tag{2.169}$$

for all $\hat{s} \geq 0$. Defining

$$f(\hat{s}) := \int_{\hat{s}}^{\infty} \int \hat{u}^4 \, d\hat{x} d\hat{t} \tag{2.170}$$

and calculating

$$f'(\hat{s}) = -\int \hat{u}^4(\hat{x}, \hat{s}) \, d\hat{x}, \tag{2.171}$$

we get the differential inequality

$$\begin{aligned} f(\hat{s}) &= \int_{\hat{s}}^{\infty} \int \hat{u}^4 \, d\hat{x} d\hat{t} \\ &\overset{(2.169)}{\leq} 6\left(\int \hat{u}^2(\hat{x}, \hat{s}) \, d\hat{x}\right)^{\frac{3}{2}} \\ &\overset{(A.2)}{\leq} 6 \, \|1\|_{L^2}^{\frac{3}{2}} \, \|\hat{u}^2\|_{L^2}^{\frac{3}{2}} \\ &= 6\left(\int \hat{u}^4(\hat{x}, \hat{s}) \, d\hat{x}\right)^{\frac{3}{4}} \\ &\overset{(2.171)}{=} 6(-f'(\hat{s}))^{\frac{3}{4}} \end{aligned}$$

such that

$$6^{-\frac{4}{3}} \leq \frac{-f'(\hat{s})}{f^{\frac{3}{4}}(\hat{s})} \tag{2.172}$$

and therefore

$$\left(f^{-\frac{1}{3}}(\hat{s})\right)' = -\frac{1}{3} \frac{f'(\hat{s})}{f^{\frac{4}{3}}(\hat{s})} \overset{(2.172)}{\geq} 6^{-\frac{4}{3}} 3^{-1} = c.$$

Integrating this yields

$$f^{-\frac{1}{3}}(\hat{s}) \geq c\hat{s}$$

so that

$$\int_{\hat{s}}^{\infty} \int \hat{u}^4 \, d\hat{x} d\hat{t} \overset{(2.170)}{=} f(\hat{s}) \leq \frac{c}{\hat{s}^3}. \tag{2.173}$$

Via Burgers' equation and the periodicity of $\hat{u}$ we get

$$\frac{d}{dt} \int \hat{u}^2 \, d\hat{x} = \int \left(\hat{u}^2\right)_{\hat{t}} \, d\hat{x} \overset{(\mathrm{EC}_0)}{\leq} -\frac{2}{3} \int \left(\hat{u}^3\right)_{\hat{x}} \, d\hat{x} = -\frac{2}{3} \hat{u}^3 \Big|_0^1 = 0 \tag{2.174}$$

such that $\int \hat{u}^2 \, d\hat{x}$ is non-increasing in time and therefore

$$\begin{aligned}
\left(\int \hat{u}^2(\hat{x}, 2\hat{s}) \, d\hat{x} \right)^2 &= \frac{1}{\hat{s}} \int_{\hat{s}}^{2\hat{s}} \left(\int \hat{u}^2(\hat{x}, 2\hat{s}) \, d\hat{x} \right)^2 d\hat{t} \\
&\overset{(2.174)}{\leq} \frac{1}{\hat{s}} \int_{\hat{s}}^{2\hat{s}} \left(\int \hat{u}^2(\hat{x}, \hat{t}) \, d\hat{x} \right)^2 d\hat{t} \\
&\overset{(A.2)}{\leq} \frac{1}{\hat{s}} \int_{\hat{s}}^{2\hat{s}} \|1\|^2_{L^2(0,1)} \, \|\hat{u}^2(\cdot, \hat{t})\|^2_{L^2(0,1)} \, d\hat{t} \\
&= \frac{1}{\hat{s}} \int_{\hat{s}}^{2\hat{s}} \int \hat{u}^4 \, d\hat{x} d\hat{t} \\
&\leq \frac{1}{\hat{s}} \int_{\hat{s}}^{\infty} \int \hat{u}^4 \, d\hat{x} d\hat{t} \\
&\overset{(2.173)}{\leq} \frac{c}{\hat{s}^4}.
\end{aligned}$$

Since this holds for all $\hat{s} \geq 0$ we conclude

$$\int \hat{u}^2(\hat{x}, \hat{t}) \, d\hat{x} \leq 4 \frac{c}{\hat{t}^2} = \frac{c_0}{\hat{t}^2}.$$

∎

2.2.5 Energy Bounds for the Kuramoto-Sivashinsky Equation

Having derived a bound for $\hat{u}$, the rescaled solution of the Kuramoto-Sivashinsky equation, we can now translate this back to the original solution.

▶ **Corollary 2.22.**
Let u be a solution of (KS), (PC_L), with initial condition (IC). Then for all $T > 2$ there exists a $L_0(T)$ such that for all $L > L_0$

$$\int_{t-1}^{t} \int u^2 \, dxdt \lesssim \frac{L^3}{(t-1)^2} \tag{2.175}$$

holds for all $t \in (2, T]$. ◀

Proof.
We prove this lemma by contradiction. Assume there exists $T > 2$ such that for all $\nu \in \mathbb{N}$ there exists L_ν and a solution u_ν of the L_ν-periodic initial value problem (KS_ν), (PC_{L_ν}), (IC_ν) such that $L_\nu \to \infty$ and

$$\int_{t_\nu-1}^{t_\nu} \int u_\nu^2 \, dxdt > c \frac{L_\nu^3}{(t_\nu-1)^2} \tag{2.176}$$

for all constants $c > 0$ and some $t_\nu \in (2, T]$. Then this also holds for $c = c_0$ as in Lemma 2.21. Since t_ν is a bounded sequence, there exists a subsequence t_{ν_k}, which we can relabel as t_ν, and $\bar{t} \in [2, T]$ such that $t_\nu \to \bar{t}$. The rescaling (2.90) and the convergence (2.123) imply

$$\hat{u}_\nu(\hat{t}, \hat{x}) = \frac{1}{L_\nu} u_\nu(\hat{t}, L_\nu \hat{x}) \to \hat{u}(\hat{t}, \hat{x})$$

in $L^4_{loc}((0, \infty) \times [0, 1])$. Hölder inequality (see Proposition A.2 in the Appendix) yields

$$\begin{aligned}
\|\hat{u}_\nu - \hat{u}\|^2_{L^2_{loc}} &= \|(\hat{u}_\nu - \hat{u})^2\|_{L^1_{loc}} \\
&\overset{(A.2)}{\leq} \|1\|_{L^2_{loc}} \|(\hat{u}_\nu - \hat{u})^2\|_{L^2_{loc}} \\
&\leq c \|\hat{u}_\nu - \hat{u}\|^2_{L^4_{loc}}
\end{aligned}$$

such that

$$\hat{u}_\nu(\hat{t},\hat{x}) = \frac{1}{L_\nu} u_\nu(\hat{t}, L_\nu \hat{x}) \to \hat{u}(\hat{t},\hat{x})$$

in $L^2_{\text{loc}}((0,\infty)\times[0,1])$. So on one hand by the assumption (2.176) we have

$$\begin{aligned}
\int_{\bar{t}-1}^{\bar{t}} \int \hat{u}^2 \, d\hat{x} d\hat{t} &= \lim_{\nu\to\infty} \int_{t_\nu-1}^{t_\nu} \int \hat{u}_\nu^2 \, d\hat{x} d\hat{t} \\
&= \lim_{\nu\to\infty} \frac{1}{L_\nu^3} \int_{t_\nu-1}^{t_\nu} \int u^2 \, dx dt \\
&\overset{(2.176)}{\geq} \lim_{\nu\to\infty} c_0 \frac{1}{(t_\nu - 1)^2} \\
&= c_0 \frac{1}{(\bar{t}-1)^2}.
\end{aligned} \tag{2.177}$$

By Theorem 2.16 $\hat{u}$ solves

$$\begin{aligned}
\hat{u}_{\hat{t}} + \left(\frac{1}{2}\hat{u}^2\right)_{\hat{x}} &= 0, \\
\left(\frac{1}{2}\hat{u}^2\right)_{\hat{t}} + \left(\frac{1}{3}\hat{u}^3\right)_{\hat{x}} &\leq \frac{1}{4}\hat{u}^2
\end{aligned}$$

in $\mathcal{D}^\star((0,\infty)\times\mathbb{R})$. $\hat{\mu} = \frac{1}{4}\hat{u}^2$ fulfills (2.131) since for bounded A with $B_r \subset A$ the Lebesgue measure $\mathcal{L}(A)$ of A is non-zero and therefore, because of $\hat{u} \in L^4_{\text{loc}}$,

$$\begin{aligned}
\frac{\mu(B_r)}{r} &= \frac{1}{r} \int_{B_r} \frac{1}{4} \hat{u}^2(\hat{x},\hat{t}) \, d\hat{x} d\hat{t} \\
&= \frac{1}{4r\mathcal{L}(A)} \int_A \int_{B_r} \hat{u}^2(\hat{x},\hat{t}) \, d\hat{x} d\hat{t} \, d\hat{y} d\hat{\tau} \\
&\leq \frac{1}{4r\mathcal{L}(A)} \int_A \int_{B_r} \hat{u}^2(\hat{y},\hat{\tau}) \, d\hat{x} d\hat{t} \, d\hat{y} d\hat{\tau} \\
&\leq \frac{1}{4r\mathcal{L}(A)} \int_{B_r} d\hat{x} d\hat{t} \int_A \hat{u}^2(\hat{y},\hat{\tau}) \, d\hat{y} d\hat{\tau}
\end{aligned}$$

$$= \frac{\pi r}{4\mathcal{L}(A)} \|\hat{u}^2\|^2_{L^2(A)}$$
$$\xrightarrow{r \to 0} 0.$$

So by Corollary 2.20 $\hat{u}$ is an entropy solution of Burgers' equation, which implies that it fulfills the requirements of Lemma 2.21. Applying Lemma 2.21 we find

$$\int \hat{u}^2(\hat{x}, \hat{t})\, d\hat{x} \le \frac{c_0}{\hat{t}^2} \tag{2.178}$$

and therefore

$$\int_{\bar{t}-1}^{\bar{t}} \int \hat{u}^2\, d\hat{x} d\hat{t} \overset{(2.178)}{\le} c_0 \int_{\bar{t}-1}^{\bar{t}} \frac{1}{\hat{t}^2} d\hat{t} = c_0 \left(\frac{1}{\bar{t}-1} - \frac{1}{\bar{t}} \right) = c_0 \frac{1}{\bar{t}(\bar{t}-1)}. \tag{2.179}$$

Since $\bar{t} \in [2, T]$ we have

$$(\bar{t} - 1)^2 = \bar{t}^2 - \bar{t} - \bar{t} + 1 < \bar{t}(\bar{t} - 1) \tag{2.180}$$

such that by (2.177) and (2.179) we get the following contradiction

$$\int_{\bar{t}-1}^{\bar{t}} \int \hat{u}^2\, d\hat{x} d\hat{t} \overset{(2.179)}{\le} c_0 \frac{1}{\bar{t}(\bar{t}-1)} \overset{(2.180)}{<} c_0 \frac{1}{(\bar{t}-1)^2} \overset{(2.177)}{\le} \int_{\bar{t}-1}^{\bar{t}} \int \hat{u}^2\, d\hat{x} d\hat{t},$$

which concludes the proof. ■

▶ **Corollary 2.23.**
There exists a universal constant $\bar{c} > 0$ such that for all $T > 0$ there exists $L_0(T)$ such that for all $L > L_0$ and any solution u of (KS), (PC_L) with initial condition (IC)

$$\int_0^L u^2(x, t)\, dx \le \bar{c} \frac{L^3}{t^2} \tag{2.181}$$

holds for all $t \in [0, T]$. ◀

Proof.
For $t \leq 2$ we have by the initial layer estimate (see Lemma 2.14)

$$\int u^2(x,t)\,dx \overset{(2.60)}{\leq} cL^3\left(1+\frac{1}{t^2}\right) \leq cL^3\frac{5}{t^2} \leq \bar{c}\frac{L^3}{t^2}.$$

For $t > 2$ we have

$$4(t-1)^2 = t^2 + 3t^2 - 8t + 4 = t^2 + (3t-2)(t-2) \geq t^2 \tag{2.182}$$

such that by Inequality (2.31) and Corollary 2.22 we get the claim as

$$\begin{aligned}
\int u^2(x,t)\,dx &= \int_{t-1}^{t}\int u^2(x,t)\,dxds \\
&\leq \int_{t-1}^{t}\sup_{\tau\in(s,t)}\int u^2(x,\tau)\,dxds \\
&\overset{(2.31)}{\lesssim} \int_{t-1}^{t} e^{\frac{t-s}{2}}\int u^2(x,s)\,dxds \\
&\leq e^{\frac{1}{2}}\int_{t-1}^{t}\int u^2(x,s)\,dxds \\
&\overset{(2.175)}{\lesssim} \frac{L^3}{(t-1)^2} \\
&\overset{(2.182)}{\lesssim} \frac{L^3}{t^2}.
\end{aligned}$$

■

By translation invariance in time we can extend this to large times.

▸ Corollary 2.24.
There exists a universal constant $\tilde{c} > 0$ such that for all $T > 0$ there exists $L_0(T)$ such that for any solution u of (KS), (PC_L) with initial condition (IC)

$$\sup_{t>T}\int_0^L u^2(x,t)\,dx \leq \tilde{c}\frac{L^3}{T^2} \tag{2.183}$$

holds for all $L > L_0$. ◂

Proof.
By translation invariance in time (see Lemma 2.10) Corollary 2.23 also applies to $v(x,t) = u(x,t+\tau)$. So for all $\kappa > 0$ and $\tau = \kappa + \frac{T}{2}$ and $t \in \left[\frac{T}{2}, T\right]$ one gets

$$\int u^2(x,t+\tau)\,dx = \int v^2(x,t)\,dx \overset{(2.181)}{\leq} \tilde{c}\frac{L^3}{t^2} \leq \tilde{c}\frac{L^3}{T}$$

and by the choice of τ

$$\int u^2(x,t)\,dx \leq \tilde{c}\frac{L^3}{T}$$

for $t \in \left[\frac{T}{2}+\tau, T+\tau\right] = \left[\kappa+T, \kappa+\frac{3}{2}T\right]$. For all $t > T$ we find κ such that $t \in \left[\kappa+T, \kappa+\frac{3}{2}T\right]$ so that

$$\sup_{t>T}\int u^2(x,t)\,dx \leq \tilde{c}\frac{L^3}{T^2}.$$

■

Now we can prove the bound claimed in (1.3) and (1.15).

▶ **Corollary 2.25.**
Let u solves (KS), ($\mathrm{PC_L}$), (IC), then

$$\limsup_{t\to\infty}\|u\|_{L^2[-0,L]} \leq o\left(L^{\frac{3}{2}}\right).$$

◀

Proof.
Corollary 2.24 yields the claim since

$$\begin{aligned}\lim_{L\to\infty} L^{-\frac{3}{2}}\limsup_{t\to\infty}\|u\|_{L^2[-0,L]} &= \lim_{L\to\infty} L^{-\frac{3}{2}}\limsup_{t\to\infty}\left(\int u^2(x,t)\,dx\right)^{\frac{1}{2}}\\ &= \lim_{L\to\infty} L^{-\frac{3}{2}}\lim_{T\to\infty}\sup_{t>T}\left(\int u^2(x,t)\,dx\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&\overset{(2.183)}{\leq} \lim_{L\to\infty} L^{-\frac{3}{2}} \lim_{T\to\infty} \left(\tilde{c}\frac{L^3}{T^2}\right)^{\frac{1}{2}} \\
&= \lim_{L\to\infty} \lim_{T\to\infty} \frac{\tilde{c}^{\frac{1}{2}}}{T} \\
&= 0.
\end{aligned}
$$

■

Appendix

▶ **Definition A.1 (Fractional Derivative).**
Based on [Ott09, Definition 2] we define for $\alpha \in \mathbb{R}$ the α-fractional derivative $|\partial_x|^\alpha u$ of u via its Fourier series $\hat{u}$ through

$$\widehat{|\partial_x|^\alpha u}(q) = |q|^\alpha \hat{u}(q). \tag{A.1}$$

◀

Some useful identities that can be found in Evans' *Partial Differential Equations* [Eva98].

▶ **Proposition A.2 (Hölder Inequality).**
For $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, $f \in L^p(U)$ and $g \in L^q(U)$

$$\|fg\|_{L^1(U)} \leq \|f\|_{L^p(U)} \|g\|_{L^q(U)}. \tag{A.2}$$

◀

▶ **Proposition A.3 (Cauchy Inequality).**
For $a, b \in \mathbb{R}$ and all $c > 0$

$$2ab \leq ca^2 + \frac{1}{c} b^2. \tag{A.3}$$

◀

▶ **Proposition A.4 (Jensen Inequality for Bounded Domains).**
Assume $f : \mathbb{R} \to \mathbb{R}$ is convex and $U \subset \mathbb{R}^n$ is open and bounded. Let $u : U \to \mathbb{R}$ be summable, then

$$f\left(\frac{1}{\mathcal{L}(U)} \int_U u \, dx \right) \leq \frac{1}{\mathcal{L}(U)} \int_U f(u) \, dx. \tag{A.4}$$

◀

▶ Proposition A.5 (Jensen Inequality for Unbounded Domains).
Assume $f : \mathbb{R} \to \mathbb{R}$ is convex, $U \subset \mathbb{R}^n$ is open and $\rho \geq 0$ such that $\int_U \rho = 1$, then

$$f\left(\int_U \varphi\rho \, dx\right) \leq \int_U f(\varphi)\rho \, dx. \tag{A.5}$$

◀

▶ Proposition A.6 (Gagliardo-Nirenberg Type Sobolev embedding).
Let U be an open subset of $\mathbb{R}^n$. Assume $1 \leq p < n$ and $u \in W_0^{1,p}(U)$, then $u \in L^{p^\star}$, where $p^\star = \frac{np}{n-p}$ and there exists a constant c, only depending on p and n, such that

$$\|u\|_{L^{p^\star}(U)} \leq c\|Du\|_{L^p(U)}. \tag{A.6}$$

◀

▶ Proposition A.7 (Morrey Type Sobolev embedding).
Let U be a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary. Assume $n < p \leq \infty$ and $u \in W^{1,p}(\bar{U})$, then, after possibly being redefined on a measure zero subset, $u \in C^{0,1-\frac{n}{p}}$ and there exists a constant c, only depending on p, n and U, such that

$$\|u\|_{C^{0,1-\frac{n}{p}}(\bar{U})} \leq c\|u\|_{W^{1,p}(U)}. \tag{A.7}$$

◀

▶ Proposition A.8 (Arzelà-Ascoli theorem).
Suppose f_k is a sequence of uniformly equicontinuous, real-valued functions defined on $\mathbb{R}^n$ and there exists a constant c such that

$$|f_k(x)| \leq c$$

for all k and $x \in \mathbb{R}^n$, then there exists a subsequence f_{k_j} and a continuous function f such that

$$f_{k_j} \to f \tag{A.8}$$

uniformly on compact subsets of $\mathbb{R}^n$. ◀

▶ **Proposition A.9 (Differential Grönwall Inequality).**
For a absolutely continuous function $\eta \geq 0$ on $[0, T]$ that satisfies

$$\eta'(t) \leq \varphi(t)\eta(t) + \psi(t)$$

with non-negative, summable functions φ, ψ on $[0, T]$

$$\eta(t) \leq e^{\int_0^t \varphi(s)\, ds}\left(\eta(0) + \int_0^t \psi(s)\, ds\right) \tag{A.9}$$

holds for all $t \in [0, T]$. ◀

Similar to the proof of Proposition A.9 in [Eva98] we prove another version of Grönwall Inequality for not necessary non-negative φ.

▶ **Proposition A.10 (Differential Grönwall Inequality).**
For a absolutely continuous function $\eta \geq 0$ on $[0, T]$ that satisfies

$$\eta'(t) \leq \varphi(t)\eta(t) + \psi(t)$$

with summable functions φ, ψ on $[0, T]$, where ψ is non-negative

$$\eta(t) \leq e^{\int_0^t \varphi(s)\, ds}\left(\eta(0) + \int_0^t e^{-\int_0^s \varphi(r)\, dr}\psi(s)\, ds\right) \tag{A.10}$$

holds for all $t \in [0, T]$. ◀

Proof.
Calculating

$$\frac{d}{ds}\left(\eta(s)e^{-\int_0^s \varphi(r)\, dr}\right) = e^{-\int_0^s \varphi(s)\, dr}(\eta'(s) - \varphi(s)\eta(s)) \leq e^{-\int_0^s \varphi(r)\, dr}\psi(s)$$

for a.e. $0 \leq s \leq T$ implies

$$\begin{aligned}\eta(t)e^{-\int_0^t \varphi(r)\, dr} &= \eta(0) + \int_0^t \frac{d}{ds}\left(\eta(s)e^{-\int_0^s \varphi(r)\, dr}\right)ds \\ &\leq \eta(0) + \int_0^t e^{-\int_0^s \varphi(r)\, dr}\psi(s)\, ds\end{aligned}$$

for all $0 \leq t \leq T$ such that

$$\eta(t) \leq e^{\int_0^t \varphi(r)\, dr}\left(\eta(0) + \int_0^t e^{-\int_0^s \varphi(r)\, dr}\psi(s)\, ds\right).$$

■

The following compactness theorems can be found in Brezis' *Functional Analysis, Sobolev Spaces and Partial Differential Equations* [Bre11, Theorem 3.18 and Corollary 3.30].

▶ **Proposition A.11 (Sequential Weak Compactness).**
Every bounded sequence in a reflexive Banach space has a weakly converging subsequence. ◀

▶ **Proposition A.12 (Sequential Weak Star Compactness).**
Every bounded sequence in the dual of a separable Banach space has a weak star convergent subsequence. ◀

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Hamburg, September 23, 2020

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