Math 742
Methods of Applied Mathematics II
Partial Differential Equations

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Contents

Lectures

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Mostly based on Evans 2010.

Chapter A

Introduction

Lecture 1 (January 05)

1 Organization

1.1 General

- Small group, grad course
  - interaction
  - discussion
  - background
  - feedback
- Blackboard or tablet?
- Lecture notes on Avenue
- This is mostly based on Evans (2010) – Partial Differential Equations
- 4 homework assignments (15% each)
  - ≈ Jan 19 → Jan 26
  - ≈ Feb 2 → Feb 9
  - ≈ Feb 16 → Mar 2 (Reading Week)
  - ≈ Mar 9 → Mar 16
- Final presentation (40%)
  - Topic selection end of February/beginning of March
  - Presentations during last lectures
    - ⋆ Mar 30
    - ⋆ Apr 1
    - ⋆ Apr 6

1.2 Roadmap

1. Classical theory
   - Transport equation
   - Laplace/Poisson equation
   - Heat equation
   - Wave equation
2. Sobolev spaces
3. Nonlinear first order PDEs
   - Conservation laws
   - Shocks
4. Second order ellipctic PDEs
5. Advanced topics
   - Calculus of variations
     - Euler-Lagrange Equations
     - Optimization Problems
   - Fluid dynamics
     - Navier-Stokes equations
     - Euler equations

2 Notation

In what follows we will try to follow the notation of Evans 2010 as closely as possible to avoid confusion.

- $U \subset \mathbb{R}^n$ is an open, i.e. U subset of the n-dimensional real Euclidean space. It is called the domain. Open to be able to define derivatives at every point $x \in U$. In the literature this is sometimes denoted by Ω. Later we will sometimes use $n+1$ dimensions to distinguish time.
- Unknown or solution $u : U \to \mathbb{R}$.
- We will also consider $u : U \to \mathbb{R}^m$ and try to be consistent to use u^k for the k-th component of u. Evans 2010 uses $\boldsymbol{u}$ but since this is inconvenient for writing we will try to specify if we are working with scalar or vector valued partial differential equations.
- $u_i = u_{x_i} = \partial_i u = \partial_{x_i} u = \frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x+he_i)-u(x)}{h}$, where e_i is the unit vector in x_i direction, is the partial derivative in x_i-direction, provided it exists. Sometimes we also write x, y, z instead of x_1, x_2, x_3.

- $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\alpha_i \in \mathbb{N}_0$ is a multiindex of order $|\alpha| = \alpha_1 + \dots + \alpha_n$.
- $D^\alpha u(x) = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}} u(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(x)$
- $\nabla^k u(x) = D^k u(x) = \{D^\alpha u \,|\, |\alpha| = k\}$ is the set of all partial derivatives of order k. For scalar u it can be seen as a point in $\mathbb{R}^{n^k}$ or as a rank k tensor. For vector valued u it is a point in $\mathbb{R}^{m\,n^k}$
- $\nabla u = Du = (\partial_1 u, \dots, \partial_n u)$ is the gradient
- $D^2 u = \begin{pmatrix} \partial_1^2 u & \dots & \partial_1 \partial_n u \\ \vdots & \ddots & \vdots \\ \partial_n \partial_1 u & \dots & \partial_n^2 u \end{pmatrix}$ is the Hessian (matrix) of u
- $\Delta u = \sum_{i=1}^n \partial_i^2 u$ is the Laplacian of u. In the literature Δu is sometimes denoted by $\nabla^2 u$.

Definition 1
An expression of the form

$$F\left(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x\right) = 0 \tag{2.1}$$

for all $x \in U$, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

is given is called a partial differential equation (PDE) of order k and $u : U \to \mathbb{R}$ is the unknown or solution.

Definition 2
Similarly

$$F\left(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x\right) = 0,$$

where

$$F : \mathbb{R}^{m\,n^k} \times \mathbb{R}^{m\,n^{k-1}} \times \dots \times \mathbb{R}^{m\,n} \times \mathbb{R}^m \times U \to \mathbb{R}^m$$

for $u : U \to \mathbb{R}^m$ is given is called a k-th order system of partial differential equationsor vector valued PDE.

3 Classification

Definition 3
A PDE is called

1. linear if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x) \tag{3.1}$$

   for given functions a_α, f. It is called homogeneous if $f \equiv 0$.

2. semilinear if it has the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0$$

3. quasilinear if it has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0$$

4. fully nonlinear if it depends nonlinearly on the highest order of derivatives.

Formally (3.1) is equivalent to

$$Lu = f, \qquad L = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha,$$

i.e. a linear differential operator L acting on u. Similarly one can define nonlinear differential operators but the terminology does not match other definitions of quasi or semilinearity. The principle part of $L = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha$ is given by $L_0 = \sum_{|\alpha|=k} a_\alpha(x) D^\alpha$. It determines many of the properties of the solutions.

Definition 4
For a general second order linear PDE in two dimensions

$$(\partial_x, \partial_y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} (\partial_x, \partial_y)\, u + (d, e)\, (\partial_x, \partial_y)\, u \overset{\text{if } a,b,c \text{ constant}}{=}$$

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y = f \tag{3.2}$$

with $(a, b, c, d, f)(x, y)$ there is an important characterization by the principle part, i.e the matrix. (3.2) is called

- elliptic if the eigenvalues have the same sign, i.e.

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = ac - b^2 > 0$$

- parabolic if one of the eigenvalues is zero, i.e.

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = ac - b^2 = 0$$

- hyperbolic if the eigenvalues have the opposite signs, i.e.

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = ac - b^2 < 0$$

There are classifications for non-second order equations which reflect the properties of the solutions.

4 Some PDEs

This should give a little motivation as to why PDEs are a big topic in mathematics and what to expect

- The Cauchy-Riemann system

$$\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v$$

  is a first order system of linear PDEs in $n = 2$ dimensions. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ the solutions $(u, v) : U \to \mathbb{R}^2$ correspond, assuming they are sufficiently smooth, to the real and imaginary part of a holomorphic function $h : U \to \mathbb{C}$.

- The Laplace's equation

$$\Delta u = 0$$

  is a homogeneous second order linear PDE.

- The Poisson's equation is given by

$$-\Delta u = f$$

  for a given f independent of u is an inhomogeneous second order linear PDE. u represents the electric potential of a charge distribution f.

- The Nonlinear Poisson equations is given by

$$-\Delta u = f(u)$$

- The linear transport equation

$$u_t + b \cdot \nabla u + cu = 0$$

  in $(t, x) \in U \subset \mathbb{R}^{n+1}$. Here we split the "special time" variable t and the gradient only acts on the spatial variable $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$ This represents a distribution u transported in direction $b \in \mathbb{R}^n$ with a source/sink c.

- The Heat or Diffusion Equation

$$u_t - \Delta u = 0$$

  describes the time evolution of diffusive quantities like heat.

- Wave equation

$$u_{tt} - \Delta u = 0$$

  describes the time evolution of waves/oscillations

- Inviscid Burger's equation

$$u_t + uu_x = 0$$

is a prototype of a nonlinear equation creating shocks.

- Schrödinger equation

$$iu_t + \Delta u = cu$$

describes the behaviour of the wave functions in quantum mechanics.

- Maxwell's equations

$$\begin{aligned}\nabla \cdot E &= f\\ \nabla \cdot B &= 0\\ E_t &= \nabla \times B + c\\ B_t &= -\nabla \times E\end{aligned}$$

describe electromagnetism.

- Navier-Stokes ($\nu > 0$) and Incompressible Euler equations ($\nu = 0$)

$$\begin{aligned}u_t + u \cdot \nabla u + \nabla p - \nu \Delta u &= 0\\ \nabla \cdot u &= 0\end{aligned}$$

describe fluid dynamics.

5 Studying PDEs

Ideally we would like to find explicit solutions of a PDE, i.e. a function u satisfying (2.1) under some constraints, for example initial and boundary conditions or regularity assumptions. However, as can be seen from the vast variety of PDEs this is always possible. Then we try to prove the existence of a solution and other properties such as uniqueness, regularity, or decay.

In fact even this is challenging. One of the only rather general results in the theory of PDEs is the Cauchy-Kovalevskaya Theorem (Evans 2010, 4.6.3 Theorem 2; Folland 1995, Chapter 1.D Theorem 1.25), which basically states the following. Every PDE of type

$$\begin{aligned}\partial_t^k u = F\left(x, t, u, (\partial_t^j \partial_x^\alpha u)_{|\alpha|+j\leq k, j<k}\right)\\ + \text{ initial conditions}\end{aligned}$$

where F and the initial conditions are analytic has a local unique analytic solution. While this seems rather general, it only yields existence in an ε neighbourhood of the initial time.

Lecture 2 (January 07)

Therefore, in the following we will first look at prototypes of different kinds of PDEs and then generalize the ideas.

1. We want to prove well-posedness, i.e.
   - the existence of solutions
   - the uniqueness of solutions
   - continuous dependence on the data given
2. The regularity of solutions plays a crucial role. In fact, what do we require for a solution? Does it have to satisfy the PDE in a classical sense, or can we relax it to an averaged sense. How do we solve PDEs if the PDE naturally develops jumps and the differential operator does not make sense anymore?
3. What additional properties do solutions satisfy?

Chapter B

Classical Theory

6 Linear Transport Equation

The linear transport equation is given by

$$u_t + b \cdot \nabla u = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty)$$

where $b \in \mathbb{R}^n$ and the unknown is $u : \mathbb{R}^n \to \mathbb{R}$. Here x represents a point in space and $t \geq 0$ time.

To find a solution notice that if we let x, t depend on some variable s and define $z(s) = u(x + y(s), t + \tau(s))$, then

$$\begin{aligned} \frac{d}{ds} z(s) = \frac{d}{ds} u(x + y(s), t + \tau(s)) &= \sum_{i=1}^{n} u_i \frac{d}{ds} y^i(s) + u_t \frac{d}{ds} \tau \\ &= \dot{y}(s) \cdot \nabla u + \dot{\tau}(s) u_t, \end{aligned}$$

where $\dot{f}(s) = \frac{d}{ds} f(s)$. If $\dot{y}(s) = b$ and $\dot{\tau}(s) = 1$ then

$$\frac{d}{ds} \left(u(x + bs, t + s) \right) = \frac{d}{ds} z(s) = b \cdot \nabla u + u_t = 0, \tag{6.1}$$

i.e. u is constant on lines in direction $(b, 1)$ in the (x, t) plane. So if we know the value at any point of the line we know it everywhere on the line. draw picture!

6.1 The Initial Value Problem

Consider

$$u_t + b \cdot \nabla u = 0 \qquad \text{for all } (x, t) \in \mathbb{R}^n \times (0, \infty) \tag{6.2}$$

$$u = g \qquad \text{for all } (x, t) \in \mathbb{R}^n \times \{t = 0\} \tag{6.3}$$

where $b \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given and g is sufficiently smooth (see Remark 5 later).

From the previous considerations the line through (x,t) in direction $(b,1)$ is given by $(x+bs, t+s)$ and it hits the plane $\mathbb{R}^n \times \{t=0\}$ at $s=-t$, i.e. $(x-bt, 0)$. So by (6.1) u is constant on this line and

$$\begin{aligned} u(x,t) &= u(x+bs, t+s)|_{s=0} = u(x+bs, t+s)|_{s=-t} \\ &= u(x-bt, 0) = g(x-bt) \end{aligned}$$

for all $x, t \in \mathbb{R}^n \times (0,\infty)$.

Remark 5 (Regularity Considerations)

1. By sufficiently smooth we mean that all the subsequent calculations are justified. In general it is rather common to consider the smooth scenario first and afterwards think about the smoothness depending on the data or what goes wrong if the data is not sufficiently regular.

2. In order for the PDE to make sense (all operators to be justified) u has to be differentiable, i.e. g should be C^1. But the profile of g is just transported in direction $(b,1)$, so $u(x,t) = g(x-tb)$ solves (6.2), (6.3) in some sense even if $g \notin C^1$. We will consider such *weak solutions* later.

3. In the following, if not explicitly specified, everything will be considered as sufficiently smooth.

6.2 The inhomogeneous initial value problem

Consider

$$u_t + b \cdot \nabla u = f \qquad \text{for all } (x,t) \in \mathbb{R}^n \times (0,\infty) \tag{6.4}$$

$$u = g \qquad \text{for all } (x,t) \in \mathbb{R}^n \times \{t=0\} \tag{6.5}$$

where $b \in \mathbb{R}^n$, $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$, and $g : \mathbb{R}^n \to \mathbb{R}$ are given.

Inspired by the previous considerations, let

$$z(s) = u(x+sb, t+s)$$

then

$$\dot{z}(s) = b \cdot \nabla u(x+sb, t+s) + u_t(x+sb, t+s) = f(x+sb, t+s).$$

Therefore

$$\begin{aligned} u(x,t) - u(x-bt, 0) = z(0) - z(-t) &= \int_{-t}^{0} \dot{z}(s)\ ds \\ &= \int_{-t}^{0} f(x+sb, t+s)\ ds = \int_{0}^{t} f(x+(\tau-t)b, \tau)\ d\tau \end{aligned}$$

and

$$\begin{aligned} u(x,t) &= u(x-bt,0) + \int_0^t f(x+(\tau-t)b,\tau)\ d\tau) \\ &= g(x-bt) + \int_0^t f(x+(\tau-t)b,\tau)\ d\tau \end{aligned}$$

solves (6.4),(6.5).

Remark 6 (Method of Characteristics)
We have solved the PDE by, in essence, letting x and t depend on a variable s which turns the PDE into an ODE in s. This describes curves in the x,t-plane called *characteristics*. Solving the ODE along these characteristics, we find the solution of the PDE.

7 Laplace's Equation

Consider Laplace's equation

$$\Delta u = 0 \tag{7.1}$$

and Poisson's equation

$$-\Delta u = f \tag{7.2}$$

The $-$ is artificial but matches the notation for general elliptic PDEs considered later. for $x \in U$, $u : \bar{U} \to \mathbb{R}$ and in (7.2) also $f : U \to \mathbb{R}$. $\Delta u = \sum_{i=1}^n \partial_i^2 u$ is called the *Laplacian* of u.

Definition 7
A function satisfying (7.1) is called a *harmonic* function.

For now consider $U = \mathbb{R}^n \backslash \{0\}$.

Laplace's equation on $\mathbb{R}^n$ is rotationally symmetric see Homework, so we look for radial solutions

$$u(x) = g(|x|).$$

For rotationally symmetric functions the Laplacian is given by see Homework

$$0 = \Delta u = g''(|x|) + \frac{n-1}{|x|} g'(|x|) = g''(r) + \frac{n-1}{r} g'(r) \tag{7.3}$$

for $x \in \mathbb{R}^n \backslash \{0\}$ or $r \in (0,\infty)$. Note that

$$\begin{aligned} \frac{d}{dr}\left(r^{n-1} g'(r)\right) &= (n-1) r^{n-2} g'(r) + r^{n-1} g'(r) \\ &= r^{n-1}\left(\frac{n-1}{r} g'(r) + g''(r)\right) \overset{(7.3)}{=} 0 \end{aligned}$$

i.e.

$$g'(r) = \tilde{c} r^{1-n}$$

and

$$g(r) = \begin{cases} c \log r + d & n = 2 \\ c r^{2-n} + d & n \geq 3 \end{cases}$$

Therefore the rotationally symmetric harmonic functions on $\mathbb{R}^n \backslash \{0\}$ are exactly

$$u(x) = \begin{cases} c \log |x| + d & n = 2 \\ c |x|^{2-n} + d & n \geq 3 \end{cases}$$

with constants $c, d \in \mathbb{R}$.

If $c \neq 0$ these functions are not singular in 0 and can not be extended continuously to $\mathbb{R}^n$. Therefore the only rotationally symmetric harmonic functions on $\mathbb{R}^n$ are constant functions.

Definition 8 (Fundamental Solution)

$$F : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}, \quad F(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\omega_n} |x|^{2-n} & n \geq 3 \end{cases}$$

where ω_n denotes the volume of the n-dimensional unit ball $B_1(0) = \{x \in \mathbb{R}^n \mid |x| < 1\}$, is called the fundamental solution of the Laplace equation.

Remark 9 (Volumes)

1. One has

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)},$$

   where Γ is the Gamma Function.

$$\omega_1 = 2 \qquad \omega_2 = \pi \qquad \omega_3 = \frac{4\pi}{3} \qquad \omega_4 = \frac{\pi^2}{2} \qquad \dots$$

2. For a general ball its volume (Lebesgue measure)

$$|B_r(x)| = \omega_n r^n$$

3. For the boundary of the n-dimensional ball, i.e. the $n-1$sphere, its surface area (Hausdorf measure) is

$$|\partial B_r(x)| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} = \omega_n n r^{n-1} = |B_r(x)| \frac{n}{r}$$

Remark 10

1. Here c is chosen such that

$$\int_{\partial B_r(0)} \nu \cdot \nabla F(x)\ dS(x) = -1 \tag{7.4}$$

   Here $B_r(0) = \{x \in \mathbb{R}^n \,|\, |x| < r\}$, the ball with radius r centered in the origin, $\partial B_r(0) = \{x \in \mathbb{R}^n \,|\, |x| = r\}$ the sphere with radius r centered in the origin, $v = \frac{x}{|x|}$, the outward unit normal.

2. If F would be smooth by divergence theorem using the dirac measure (heuristically dirac function)

$$\begin{aligned}\int_{B_r(0)} \delta_0(x)\ dx \text{"="} \int_{B_r(0)} d\delta_0(x) = 1 \overset{(7.4)}{=} -\int_{\partial B_r(0)} \nu \cdot \nabla F(x)\ dS(x) \\ = -\int_{B_r(0)} \nabla \cdot \nabla F(x)\ dx = -\int_{B_r(0)} \Delta F(x)\ dx\end{aligned}$$

   for all $r > 0$ motivating the expression "$-\Delta F = \delta_0$". Later we will make this rigorous.

3. The Poisson equation (7.2) describes a the electric potential u for a charge distribution f. Therefore the fundamental solution yields the potential of a unit point charge at 0.

When shifting the fundamental solution by y, it still solves $\Delta F(x-y) = 0$ for all $x \neq y$. Considering the Poisson equation

$$-\Delta u = f$$

Since the fundamental solution describes the potential of a unit point charge at 0, if we have multiple point charges we can construct a solution by adding finitely many shifted fundamental solutions, or try to generalize by convolution.

Theorem 11 (Solution to Poisson's equation)
Suppose $f \in C_c^2(\mathbb{R}^n)$, then

$$u(x) = \int_{\mathbb{R}^n} F(x-y) f(y)\ dy = \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log|x-y|\ dy & n = 2 \\ \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy & n \geq 3 \end{cases}$$

satisfies

1. $u \in C^2$
2. u solves $-\Delta u = f$ in $\mathbb{R}^n$.

Proof
Since $\nabla^2 F \sim |x|^{-n}$ we can not immediately differentiate. So we change variables

$$u(x) = \int_{\mathbb{R}^n} F(x-y)f(y)dy \overset{\star}{=} \int_{R^n} F(y)f(x-y)\ dy$$

In $\star$ we change $y \to x - y$ and change swap limits to get back $\mathbb{R}^n$ in the right order Note first that we can exchange differentiation and integration since

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} F(y)\frac{f(x+he_i-y)-f(x-y)}{h}dy$$

and therefore

$$\begin{aligned}
&\left|\partial_i u(x) - \int_{\mathbb{R}^n} F(y)\partial_i f(x-y)\ dy\right| \\
&\quad = \lim_{h\to 0}\left|\int_{\mathbb{R}^n} F(y)\left(\frac{f(x+he_i-y)-f(x-y)}{h} - \partial_i f(x-y)\right) dy\right| \\
&\quad \le \underbrace{\int_{\substack{\text{shifted}\\ \text{supp}(f,\partial_x f)}} |F(y)|\ dy}_{\le C} \underbrace{\lim_{h\to 0}\sup_{y\in\mathbb{R}^n}\left|\frac{f(x+he_i-y)-f(x-y)}{h} - \partial_i f(x-y)\right|}_{\to 0 \text{ since } f\in C_c^2} \\
&\quad = 0
\end{aligned}$$

For a general exchange of derivative and integration theorem see here. Analogously for the second derivative. Therefore

$$\partial_{x_i}\partial_{x_j}u(x) = \int_{R^n} F(y)\underbrace{\partial_{x_i}\partial_{x_j}f(x-y)}_{\in C^0 \text{ w.r.t } x}\ dy$$

so it is continuous in x, proving part 1.

Lecture 3 (January 12)

Because of the singularity of F we need to split the integral

$$\begin{aligned}
\Delta u &= \int_{R^n} F(y)\Delta f(x-y)\ dy \\
&= \underbrace{\int_{B_\varepsilon(0)} F(y)\Delta f(x-y)\ dy}_{=I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} F(y)\Delta f(x-y)\ dy}_{=J_\varepsilon}
\end{aligned}$$

then

$$\begin{aligned}
|I_\varepsilon| &\le \underbrace{\|\nabla^2 f\|_{L^\infty(\mathbb{R}^n)}}_{\le c \text{ is this clear?}} \int_{B_\varepsilon(0)} |F(y)|\ dy = c\omega_n \int_0^\varepsilon r^{n-1}|F(r)|\ dr \\
&\le cC\begin{cases} \varepsilon^2|\log\varepsilon| & n=2 \\ \varepsilon^2 & n\ge 3 \end{cases}
\end{aligned}$$

Therefore $\lim_{\varepsilon\to 0} I_\varepsilon = 0$.

For J_ε integration by parts yields

$$J_\varepsilon = \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} F(y)\Delta f(x-y)\ dy$$
$$= \underbrace{\int_{\partial B_\varepsilon(0)} F(y)\nu\cdot\nabla f(x-y)\ dS(y)}_{L_\varepsilon}$$
$$\underbrace{-\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \nabla F(y)\cdot\nabla f(x-y)\ dy}_{K_\varepsilon}$$

Again

$$|L_\varepsilon| \leq \|\nabla f\|_{L^\infty(\partial B_\varepsilon(0))}\int_{\partial B_\varepsilon(0)} |F(y)|\ dy \leq \begin{cases} C\varepsilon|\log\varepsilon| & n=2 \\ C\varepsilon & n\geq 3 \end{cases}$$

implying $L_\varepsilon \to 0$.

Again integration by parts for K_ε yields

$$K_\varepsilon = -\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \nabla F(y)\cdot\nabla f(x-y)\ dy$$
$$= -\int_{\partial B_\varepsilon(0)} \nabla F(y)\cdot\nu f(x-y)\ dS(y) + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \underbrace{\Delta F(y)}_{=0} f(x-y)\ dy$$

Since by definition $\nabla F(y) = \frac{-1}{n\omega_n}\frac{y}{|y|^n}$ one has

$$\begin{aligned} K_\varepsilon &= -\int_{\partial B_\varepsilon(0)} \nabla F(y)\cdot\nu f(x-y)\ dS(y) \\ &= -\int_{\partial B_\varepsilon(0)} \frac{-1}{n\omega_n}\frac{y}{|y|^n}\cdot\frac{-y}{|y|} f(x-y)\ dS(y) \\ &= -\frac{1}{n\omega_n}\int_{\partial B_\varepsilon(0)} \frac{1}{\varepsilon^{n-1}} f(x-y)dS(y) = -⨍_{\partial B_\varepsilon(0)} f(x-y)\ dS(y) \overset{\varepsilon\to 0}{\to} -f(x) \end{aligned} \tag{7.5}$$

Combining everything and letting $\varepsilon\to 0$

$$\Delta u(x) = f(x)$$

proving part 2. □

Remark 12

1. $f\in C_c^2(U)$ means $f\in C^2(U)$ with compact support, i.e its support $\mathrm{supp}(f) = \overline{\{x\in U \mid f(x)\neq 0\}}$ is compact
2. Integration by parts formula clear (Homework?) otherwise use max since $f\in C_c^2$
3. Hölder's inequality clear?

7.1 Mean Value Properties

Theorem 13 (Mean-value formula for harmonic functions)
If $u \in C^2$ is harmonic, then

$$u(x) = \fint_{\partial B_r(x)} u(y) dS(y) = \fint_{B_r(x)} u(y)\ dy$$

for all balls $B(x,r) \subset U$.

Proof
For

$$\varphi(r) = \fint_{\partial B_r(x)} u(y)\ dS(y) \overset{y=x+rz}{=} \fint_{\partial B_1(0)} u(x+rz)\ dS(z).$$

one has

$$\begin{aligned}\varphi'(r) &= \fint_{\partial B_1(0)} Du(x+rz)\cdot z\ dS(z) = \fint_{\partial B_r(x)} Du(y)\cdot \frac{y-x}{r}\ dS(y) \\ &= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} Du(y)\cdot n\ dS(y) = \frac{1}{|\partial B_r(x)|}\int_{B_r(x)} \underbrace{D\cdot D}_{=\Delta} u(y)\ dy = 0\end{aligned} \tag{7.6}$$

Therefore

$$\varphi(r) = \lim_{\rho\to 0}\varphi(\rho) = \lim_{\rho\to 0}\fint_{\partial B_\rho(x)} u(y)\ dS(y) = u(x)$$

For the volume integral note that

$$\begin{aligned}\fint_{B_r(x)} u(y)\ dy &= \frac{1}{\omega_n r^n}\int_{B_r(x)} u(y)\ dy = \frac{1}{\omega_n r^n}\int_0^r \underbrace{\int_{\partial B_\rho(x)} u(y)\ dS(y)}_{|\partial B_\rho(x)|u(x)}\ d\rho \\ &= \frac{1}{\omega_n r^n}\int_0^r \omega_n n\rho^{n-1}u(x)\ d\rho = u(x)\end{aligned}$$

□

Theorem 14 (Reverse mean value property)
If $u \in C^2$ satisfies

$$u(x) = \fint_{\partial B_r(x)} u(y)\ dS(y)$$

for all $B_r(x) \subset U$, then u is harmonic in U.

Proof
Assume $\Delta u \not\equiv 0$, then by continuity there exists some $B_r(x) \subset U$ such that $\Delta u > 0$ in $B_r(x)$. But since

$$\varphi(r) = \fint_{\partial B_r(x)} u(y)\ dS(y) = u(x)$$

is constant in r we have

$$0 = \varphi'(r) \overset{(7.6)}{=} \frac{1}{|\partial B_r(x)|} \int_{B_r(x)} \Delta u(y)\ dy > 0$$

a contradiction. □

Remark/Definition 15 (Sub-/Superharmonic Functions)

1. $u \in C^2$ is called subharmonic on U if $\Delta u \geq 0$ on U. The counterintuitive terminology will be clear in the comparison principle later The previous proof shows that for subharmonic functions

$$u(x) \leq \fint_{B_r(x)} u(y)\ dy \leq \fint_{\partial B_r(x)} u(y)\ dS(y) \tag{7.7}$$

   and if even $\Delta u > 0$

$$u(x) < \fint_{B_r(x)} u(y)\ dy < \fint_{\partial B_r(x)} u(y)\ dS(y)$$

2. Similar superharmonic for $\Delta u \leq 0$.

3. Exactly as in Theorem 14 (7.7) actually characterizes subharmonic functions.

7.2 Maximum Principles

Theorem 16 (Weak maximum principle)
Suppose U is bounded and $u \in C^2(U) \cap C(\overline{U})$ is subharmonic in U. Then

$$u(x) \leq \max_{\partial U} u$$

for all $x \in U$.

Theorem 17 (Strong maximum principle)
If U is connected and bounded and a subharmonic function $u \in C^2$ attains its global maximum in U, then u is constant.

Remark 18

1. Roughly, the weak maximum principle asserts that the maximum is attained on the boundary and the strong one asserts that it is only attained on the boundary, expect for the trivial case.

2. For superharmonic functions one has equivalent minimum principles and for harmonic functions

$$|u| \leq \max_{\partial U} |u|$$

Proof of Theorem 16.

- Since U is bounded ∂U is compact and therefore $\max_{\partial U} u$ exists.
- Assume at first $\Delta u > 0$. We will prove that there is no maximum in Ω (so even a strong principle type for strict inequality). Towards contradiction, assume that u attains a maximum in $x_0 \in U$. Then necessarily $D^2 u$ has to be semi-negative, i.e. all eigenvalues have to be non-positive. But since the trace of a matrix is the sum of its eigenvalues

$$0 \overset{\text{assumption}}{<} \Delta u(x_0) = \operatorname{tr}(D^2 u(x_0)) \leq 0,$$

a contradiction.

- If u is only subharmonic, $\Delta u \geq 0$. Let $\varepsilon > 0$ and define

$$u_\varepsilon(x) = u(x) + \varepsilon |x|^2.$$

Then

$$\Delta u_\varepsilon = \Delta u + 2\varepsilon n \geq 2\varepsilon n > 0.$$

Note that also the maximum of u_ε exists since U is bounded. So the previous part applies, stating that the maximum of u_ε is not in U, so it has to be on the boundary. Therefore

$$\begin{aligned} u(y) &\leq \max_{x \in \overline{U}} (u(x) + \varepsilon |x|^2) = \max_{x \in \overline{U}} u_\varepsilon(x) \leq \max_{x \in \partial U} u_\varepsilon(x) \\ &\leq \max_{\partial U} u + \varepsilon \underbrace{\max_{\partial U} |x|^2}_{\leq C} \end{aligned}$$

for all ε. Therefore taking the limit $\varepsilon \to 0$

$$u(y) \leq \max_{\partial U} u.$$

□

Lecture 4 (January 14)

Proof of Theorem 17.
Set $M = \max_U u \in \mathbb{R}$. Then by assumption the set $\{x \in U \mid u(x) = M\} \neq \varnothing$ and the set is closed by continuity (The preimage of a continuous function on a closed set is closed).

Next we show that it is also open. For $a \in \{x \in U \,|\, u(x) = M\}$ and $r > 0$ with $\overline{B_r(a)} \subset U$, the mean value property states that

$$M = u(a) \leq \fint_{B_r(a)} u(x)\ dx \leq \max_{y \in U} u(y) \fint_{B_r(a)} dx = M$$

Therefore equality has to hold and since u is continuous the equality shows $u \equiv M$ on all of $B_r(a)$, implying $B_r(a) \subset \{u = M\}$, i.e. $\{u = M\}$ is open.

So $\{u = M\} \subset U$ is open, closed and non-empty. Since U is connected, this implies $U = \{u = M\}$, i.e. u is constant. □

Theorem 19 (Comparison Principle)
If $u, v \in C^2(U) \cap C^0(\overline{U})$ and

$$\begin{aligned} \Delta u \geq \Delta v \qquad & \text{in } U \\ u \leq v \qquad & \text{on } \partial U \end{aligned}$$

then

$$u \leq v \qquad \text{in } \overline{U}$$

Proof
Let $w = u - v$. Then w is subharmonic, since

$$\Delta w = \Delta u - \Delta v \geq 0$$

and

$$w = u - v \leq 0$$

So by the maximum principle

$$u - v = w \leq 0$$

□

Remark 20
This is the reason $\Delta u \geq 0$ is called subharmonic. For a harmonic function h and

$$\begin{aligned} \Delta u \geq 0 = \Delta h \qquad & \text{in } U \\ u \leq h \qquad & \text{on } \partial U \end{aligned}$$

one gets

$$u \leq h \qquad \text{in } \overline{U}$$

so u is below the harmonic function.

Corollary 21 (Positivity)
Let $u \in C^2(U) \cap C^0(\overline{U})$ solve

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } U \\ u &= g \qquad \text{on } \partial U \end{aligned}$$

1. If $g \geq 0$ (or $g \leq 0$) then $u \geq 0$ (respectively $u \leq 0$) in $\overline{U}$.
2. If additionally U is connected and $g > 0$ (or $g < 0$) at some point on ∂U, then u is strictly positive (strictly negative) everywhere in U

Proof

1. by the weak maximum principle.
2. by the strong maximum principle.

□

Theorem 22 (Uniqueness)
Let $g \in C^0(\partial U)$, $f \in C^0(U)$. Then there exists at most one solution $u \in C^2(U) \cap C^0(\overline{U})$ of

$$-\Delta u = f \qquad \text{in } U \tag{7.8}$$

$$u = g \qquad \text{on } \partial U \tag{7.9}$$

Proof
If u_1 and u_2 satisfy (7.8), (7.9), then by linearity for $w = u_1 - u_2$

$$\begin{aligned} -\Delta w &= 0 \qquad \text{in } U \\ w &= 0 \qquad \text{on } \partial U \end{aligned}$$

and so the maximum principle implies $0 = w = u_1 - u_2$ in $\overline{U}$. □

Theorem 23 (Continuous Dependence on Data)
If U is bounded and has maximum width l.

$$l = \inf_l \left\{ U \subset \left\{ x \in \mathbb{R}^n \,\middle|\, |e \cdot (x-a)| \leq \frac{1}{2} l \right\} \text{ for some } a, e \in \mathbb{R}^n, \text{ with } |e| = 1 \right\}$$

If $u \in C^2(U) \cap C^0(\overline{U})$ solves

$$\begin{aligned} -\Delta u &= f \qquad \text{in } U \\ u &= g \qquad \text{on } \partial U \end{aligned}$$

and $\tilde{u} \in C^2(U) \cap C^0(\overline{U})$ solves

$$\begin{aligned} -\Delta \tilde{u} &= \tilde{f} \qquad \text{in } U \\ \tilde{u} &= \tilde{g} \qquad \text{on } \partial U \end{aligned}$$

then

$$\max_{\overline{U}} |\tilde{u} - u| \leq \max_{\partial U} |\tilde{g} - g| + \frac{1}{8} l^2 \sup_U |\tilde{f} - f|$$

Proof
By linearity we can assume that $\tilde{u} = \tilde{f} = \tilde{g} = 0$ everywhere and by translation and rotation invariance we can assume that $U \subset (-\frac{1}{2}l, \frac{1}{2}l) \times \mathbb{R}^{n-1}$.

Let $M = \sup_U |f|$ and $w(x) = u(x) + \frac{1}{2}Mx_1^2$. Then

$$\Delta w = \Delta u + M = f + M \geq 0$$

in U, i.e. w is subharmonic. Therefore

$$\max_{\overline{U}} u \overset{\text{def}}{\leq} \max_{\overline{U}} w \overset{\text{max prin}}{\leq} \max_{\partial U} w \overset{\text{def}}{\leq} \max_{\partial U} u + \frac{1}{2}M \max_{|x_1| \leq \frac{1}{2}l} x_1^2 = \max_{\partial U} g + \frac{1}{8}l^2 M.$$

Similarly $v = -u + \frac{1}{2}Mx_1^2$ yields

$$\Delta v = -f + M \geq 0$$

and

$$\begin{aligned} -\min_{\overline{U}} u = \max_{\overline{U}}(-u) &\overset{\text{def}}{\leq} \max_{\overline{U}} v \overset{\text{max prin}}{\leq} \max_{\partial U} v \overset{\text{def}}{\leq} \max_{\partial U}(-u) + \frac{1}{2}M \max_{|x_1| \leq \frac{1}{2}l} x_1^2 \\ &= \max_{\partial U}(-g) + \frac{1}{8}l^2 M = -\min_{\partial U} g + \frac{1}{8}l^2 M \end{aligned}$$

implying

$$\min_{\overline{U}} u \geq \min_{\partial U} g - \frac{1}{8}l^2 M$$

Together

$$\max_{\overline{U}} |u| \leq \max_{\partial U} |g| + \frac{1}{8}l^2 M.$$

□

Lecture 5 (January 19)

7.3 Regularity

Harmonic functions are actually smooth in the interior.

Theorem 24 (Smoothness)
If $u \in C^0(U)$ satisfies the mean value property

$$u(x) = \fint_{\partial B_r(x)} u \; dS = \fint_{B_r(x)} u \; dy$$

for every ball $B_r(x) \subset U$, then

$$u \in C^\infty(U).$$

Remark 25
u might not be smooth or continuous on the boundary.

Proof
Let η be a mollifier. Set $u_\varepsilon = \eta_\varepsilon * u$ in $U_\varepsilon = \{x \in U \,|\, \mathrm{dist}(x, \partial U) > \varepsilon\}$. Then $u_\varepsilon \in C^\infty(U_\varepsilon)$ by homework.

But also

$$\begin{aligned}
u_\varepsilon(x) &= \int_U \eta_\varepsilon(x-y)u(y)\,dy \\
&= \frac{1}{\varepsilon^n}\int_{B_\varepsilon(x)} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\ dy \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \underbrace{\left(\int_{\partial B_r(x)} u(y)\ dS(y)\right)}_{=u(x)|\partial B_r(x)|} dr \\
&= u(x)\frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)|\partial B_r(x)|\,dr \\
&= u(x)\frac{1}{\varepsilon^n}\int_{B_\varepsilon(x)} \eta\left(\frac{y}{\varepsilon}\right)\ dy \\
&= u(x)\int_{B_\varepsilon(x)} \eta_\varepsilon(y)\ dy \\
&= u(x)
\end{aligned}$$

Therefore $u = u_\varepsilon$ in U_ε and therefore $u \in C^\infty(U_\varepsilon)$ for all $\varepsilon > 0$. □

In fact one can prove even prove analyticity. For this we need the following.

Lemma 26
Let u be harmonic in U. Then

$$|\nabla^\alpha u(x_0)| \leq \frac{1}{\omega_n r^n}\left(\frac{2^{n+1}nk}{r}\right)^k \|u\|_{L^1(B_r(x_0))} \tag{7.10}$$

for all x_0 with $B_r(x_0) \subset U$ and $|\alpha| = k$.

Proof
By induction.

- $k = 0$ is the mean value theorem.

- For $k \to k+1$ note that if $\nabla^\alpha u$ is harmonic, so is $\partial_i \nabla^\alpha u$ and therefore

$$\begin{aligned} |\partial_i \nabla^\alpha u(x_0)| &= \left| \fint_{B_{\frac{r}{k+1}}(x_0)} \partial_i \nabla^\alpha u \, dx \right| \\ &= \left| \frac{(k+1)^n}{\omega_n r^n} \int_{\partial B_{\frac{r}{k+1}}(x_0)} \nu_i \nabla^\alpha u \, dS \right| \\ &\leq \frac{(k+1)n}{r} \|\nabla^\alpha\|_{L^\infty(\partial B_{\frac{r}{k+1}}(x_0))} \end{aligned}$$

and for $x \in \partial B_{\frac{r}{k+1}}(x_0)$ one has $B_{\frac{k}{k+1}r}(x) \subset B_r(x_0) \subset U$. Therefore the induction hypothesis (7.10) implies

$$\begin{aligned} |\partial_i \nabla^\alpha u(x_0)| &\leq \frac{(k+1)n}{r} \frac{1}{\omega_n \left(\frac{k}{k+1} r\right)^n} \left(\frac{2^{n+1} nk}{\frac{k}{k+1} r} \right)^k \|u\|_{L^1(B_{\frac{k}{k+1}r}(x))} \\ &= \frac{\left(\frac{k+1}{k}\right)^n n(k+1)}{r} \frac{1}{\omega_n r^n} \left(\frac{2^{n+1} n(k+1)}{r} \right)^k \|u\|_{L^1(B_{\frac{k}{k+1}r}(x))} \\ &\leq \frac{1}{\omega_n r^n} \left(\frac{2^{n+1} n(k+1)}{r} \right)^{k+1} \|u\|_{L^1(B_r(x_0))} \end{aligned}$$

(for $0 = k \to k+1 = 1$ and $x \in \partial B_{\frac{r}{2}}(x_0)$ use $\frac{1}{2}r$ instead of $\frac{k}{k+1}r$ in this last step)

□

Theorem 27
If u is harmonic in U, then u is analytic in U.

Proof
Fix a ball $\overline{B_{2r}(a)} \subset U$. Then we estimate the remainder for the Taylor series for $x \in B_r(a)$. The Lagrange estimate says (since $u \in C^\infty(U)$)

$$R_a^{k-1}(x) = u(x) - \sum_{|\alpha| \leq k-1} \frac{\nabla^\alpha u(a)}{\alpha!} (x-a)^\alpha = \sum_{|\alpha|=k} \frac{\nabla^\alpha u(a + t(x-a))}{\alpha!} (x-a)^\alpha$$

for some $0 \leq t \leq 1$. Assuming

$$|x - a| < \frac{r}{2^{n+1} en} \tag{7.11}$$

Lemma 26 yields

$$
\begin{aligned}
|R_a^{k-1}(x)| &\le \frac{|x-a|^k}{k!} \sup_{x\in B_r(a)} |\nabla^\alpha u(x)| \\
&\le \frac{|x-a|^k}{k!} \frac{1}{\omega_n (2r)^n} \left(\frac{2^{n+1} n k}{2r}\right)^k \|u\|_{L^1(B_{2r}(x_0))} \\
&\overset{(7.11)}{\le} \frac{1}{k!\omega_n r^n} \left(\frac{k}{e}\right)^k \frac{1}{2^k} \|u\|_{L^1(B_{2r(x_0)})} \\
&\le \frac{1}{\omega_n r^n} \frac{1}{2^k} \|u\|_{L^1(B_{2r(x_0)})} \\
&\overset{k\to\infty}{\to} 0
\end{aligned}
$$

where we used $(\frac{k}{e})^k < k!$ (can be proven by induction or Stirling's formula). Therefore in an arbitrary point $a \in U$ the Taylor series converges uniformly in a neighbourhood $B_{\frac{r}{2^{n+1}en}}(a)$ of a. □

7.4 Green's functions

We would like to solve

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } U && (7.12) \\
u &= g \qquad \text{on } \partial U && (7.13)
\end{aligned}
$$

for a general domain U, which here is assumed to be C^1 meaning it can be locally represented by a C^1 function.

The fundamental solution seems like a good choice, but by Green's formula see Homework for $V_\varepsilon = U - B_\varepsilon(x)$, where $B_\varepsilon(x) \subset U$ (to remove the singularity of the fundamental solution)

$$
\begin{aligned}
&\int_{V_\varepsilon} u(y) \underbrace{\Delta F(x-y)}_{=0} - F(y-x)\Delta u(x)\, dx \\
&\qquad = \int_{\partial V_\varepsilon} u(y)\nu \cdot \nabla F(y-x) - F(y-x)\nu \cdot \nabla u(y)\, dS(y)
\end{aligned}
$$

for all $u \in C^2$ and

$$
\left| \int_{\partial B_\varepsilon(x)} F(y-x)\nu \cdot \nabla u(y)\, dS(y) \right| \le C\varepsilon^{n-1} \max_{\partial B_\varepsilon(0)} |F(x)| \overset{\varepsilon\to 0}{\to} 0
$$

Previously

$$
\int_{\partial B_\varepsilon(x)} u(y)\nu \cdot \nabla F(y-x)\, dS(y) \overset{(7.5)}{=} \fint_{\partial B_\varepsilon(x)} u(y)\, dS(y) \to u(x)
$$

Hence

$$u(x) = \int_{\partial U} F(y-x)\nu \cdot \nabla u(y) - u(y)\nu \cdot \nabla F(y-x)\, dS(y)$$
$$- \int_U F(y-x)\Delta u(y)\, dy$$

for all $u \in C^2(\overline{U})$.

Now if u solves (7.12), (7.13) we almost have a solution formula but we do not know $\nu \cdot \nabla u$ on ∂U. The idea is to introduce a corrector function $\varphi^x(y)$ that solves

$$-\Delta\varphi^x = 0 \qquad \text{in } U \tag{7.14}$$
$$\varphi^x(y) = F(y-x) \qquad \text{on } \partial U \tag{7.15}$$

since this Green's formula yields

$$\begin{aligned} -\int_U \varphi^x(y)\Delta u(y)\, dy &= \int_U u(y) \underbrace{\Delta\varphi^x(y)}_{=0} - \varphi^x(y)\Delta u(y)\, dy \\ &= \int_{\partial U} u(y)\nu \cdot \nabla\varphi^x(y) - \underbrace{\varphi^x(y)}_{=F(y-x)} \nu \cdot \nabla u(y)\, dS(y) \\ &= \int_{\partial U} u(y)\nu \cdot \nabla\varphi^x(y) - F(y-x)\nu \cdot \nabla u(y)\, dS(y) \end{aligned}$$

and we can now express the final missing boundary term.

Definition 28
The function defined by

$$G(x,y) = F(y-x) - \varphi^x(y)$$

for all $x, y \in U$ with $x \neq y$ is called Green's function of U.

Lemma 29 (Green's formula)
The Green's function satisfies

$$u(x) = -\int_U G(x,y)\Delta u(y)\, dy - \int_{\partial U} u(y)\nu \cdot \nabla_y G(x,y)\, dS(y) \tag{7.16}$$

for all $u \in C^2(\overline{U})$ and if $u \in C^2(\overline{U})$ solves

$$-\Delta u = f \qquad \text{in } U$$
$$u = g \qquad \text{on } \partial U$$

then

$$u(x) = -\int_U G(x,y)f(y)\, dy - \int_{\partial U} g(y)\nu \cdot \nabla_y G(x,y)\, dS(y).$$

Remark 30

1. We have a solution formula!
2. But we need to find the corrector function φ^x, which can be hard but only depends on the domain!
3. For fixed $x \in U$ the Green's function $G(x,y)$ solves

$$\begin{aligned} -\Delta_y G &= \delta_x && \text{in } U \\ G &= 0 && \text{on } \partial U \end{aligned}$$

4. G is actually symmetric, i.e. $G(x,y) = G(y,x)$ for all $x \neq y$.

Lecture 6 (January 21)

For simple geometries we can find φ^x based on a reflection idea.

Green's function for the half space

Consider the half space $U = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$.

We know that for $x \neq y$, the fundamental solution solves $\Delta F(y-x) = 0$, so (7.14) and it matches the boundary term we want, i.e. (7.15). The only problem is the singularity, but we can "reflect" it outside the domain.

Define $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$, the reflection of x with respect to the boundary ∂R^n_+. Then for

$$\varphi^x(y) = F(y - \tilde{x}) = F(y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n + x_n)$$

we have

$$\varphi^x(y) = F(y - x) \qquad \text{on } \partial\mathbb{R}^n_+$$

due to symmetry in every component and $\Delta\varphi^x(y) = 0$ in $\mathbb{R}^n_+$ since the singularity is not in $\mathbb{R}^n_+$.

Therefore the Green's function is given by

$$G(x,y) = F(y-x) - \varphi^x(y) = F(y-x) - F(y-\tilde{x})$$

and Green's formula (7.16), i.e.

$$u(x) = -\int_{\mathbb{R}^n_+} G(x,y) f(y)\, dy - \int_{\partial\mathbb{R}^n_+} u(y)\nu \cdot \nabla_y G(x,y)\, dS(y)$$

yields a solution to the Poisson equation

$$\begin{aligned} -\Delta u &= f && \text{in } \mathbb{R}^n_+ \\ u &= g && \text{on } \partial\mathbb{R}^n_+ \end{aligned}$$

Remark 31
Similarly one can find Green's functions for stripes and cubes by (infinitely often) repeated reflection. draw picture

Green's function for the unit ball

For $U = B_1(0)$, similar to before, reflect on the boundary

$$x^\star = \frac{x}{|x|^2}$$

then

$$\varphi^x(y) = F(|x|(y - x^\star))$$

solves $\Delta\varphi^x(y) = 0$ in $B_1(x)$ and $\varphi^x(y) = F(y - x)$ since on $\partial B_1(x)$

$$|x|^2(y - x^\star)^2 = |x|^2|y|^2 - 2x \cdot y + 1 \overset{|y|=1}{=} |x|^2 - 2x \cdot y + |y|^2 = |x - y|^2.$$

Therefore the Green's function is given by

$$G(x, y) = F(y - x) - \varphi^x(y) = F(y - x) - F(|x|(y - x^\star))$$

and Green's formula (7.16), i.e.

$$u(x) = -\int_{B_1(0)} G(x, y)f(y)\, dy - \int_{\partial B_1(0)} u(y)\nu \cdot \nabla_y G(x, y)\, dS(y)$$

yields a solution to the Poisson equation

$$\begin{aligned} -\Delta u &= f \qquad \text{in } B_1(0) \\ u &= g \qquad \text{on } \partial B_1(0) \end{aligned}$$

Remark 32
For an arbitrary ball $B_r(a)$ one has

$$G_{B_r(a)}(x, y) = r^{2-n} G_{B_1(0)}\left(\frac{x - a}{r}, \frac{x - a}{r}\right)$$

The existence of solutions via these Green's functions is quite tedious, but yields an explicit formula. Later we will see that solutions to the Poisson equation exist for a quite general class of f, g, U.

Solutions to

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } u \\ u &= g \qquad \text{on } \partial U \end{aligned}$$

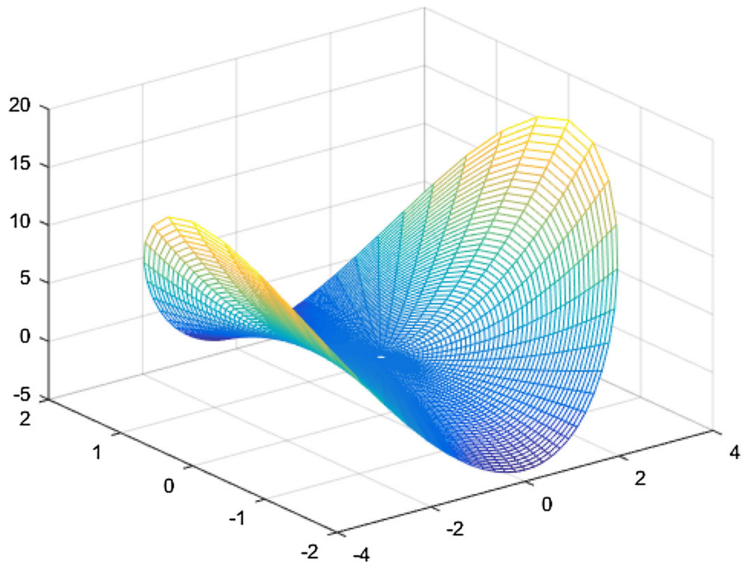

Figure B.1: $U = B_4(0)$ and $g(x,y) = x^2 - y^2$ taken from Cheruvu 2917

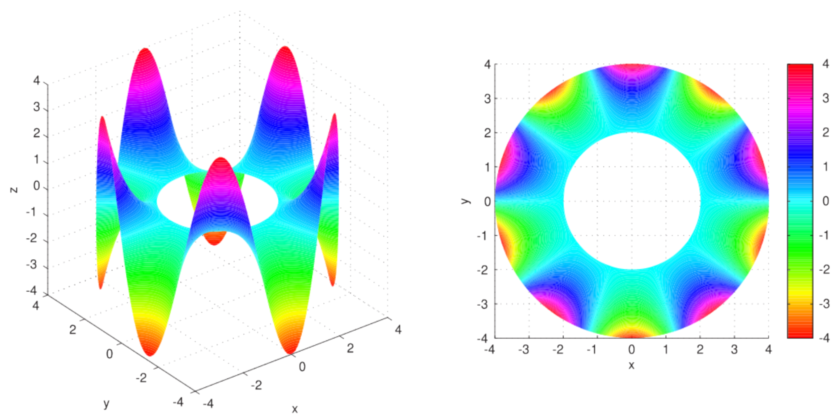

Figure B.2: $U = B_4 \setminus \overline{B_2}$ and $g(r=2) = 0$, $g(r=4) = 4\sin(5\theta)$ taken from Wikipedia - Laplace's equation

look similar to minimal surfaces which satisfy

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

8 Heat equation

The (inhomogeneous if $f \not\equiv 0$) heat equation is

$$u_t - \Delta u = f \qquad \text{in } U \times (0, \infty)$$

for some open $U \subset \mathbb{R}^n$, where $f(x,t)$ is given and $u(x,t)$ is the solution. It models diffusion processes such as heat, chemical concentration and is connected to Brownian motions.

8.1 Full space

Fundamental solution

We first want to consider $U = \mathbb{R}^n$ and $f \equiv 0$ and construct solutions similar to before. Note that for a solution u we can rescale it to $v(x,t) = u(\lambda x, \lambda^2 t)$ and for $y = \lambda x$, $\tau = \lambda^2 t$

$$\begin{aligned} v_t - \Delta v &= u_t(\lambda x, \lambda^2 t) - \Delta_x u(\lambda x, \lambda^2 t) = u_\tau(y,\tau)\lambda^2 - \Delta_y u(y,\tau)\lambda^2 \\ &= \lambda^2(u_\tau(y,\tau) - \Delta_y u(y,\tau)) = 0. \end{aligned}$$

and if $\lambda = \frac{1}{t^2}$ we get $v(x,t) = u\left(\frac{x}{t^2}, 1\right) = \tilde{v}\left(\frac{x}{t^2}\right)$ solve the heat equation.

Because it works quicker we want to look for a structure

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

for $\alpha, \beta > 0$. Then defining $y = \frac{x}{t^\beta}$

$$\begin{aligned} 0 &= u_t - \Delta u \\ &= -\alpha t^{-\alpha-1} v\left(\frac{x}{t^\beta}\right) + t^{-\alpha} \nabla_y v\left(\frac{x}{t^\beta}\right) \cdot x(-\beta) t^{-\beta-1} - t^{-\alpha} \nabla_x \cdot \left(\nabla_y v\left(\frac{x}{t^\beta}\right) t^{-\beta}\right) \\ &= -\alpha t^{-\alpha-1} v(y) - \beta t^{-\alpha-1} y \cdot \nabla_y v(y) - t^{-\alpha-2\beta} \Delta_y v(y) \end{aligned}$$

To match the t prefactors set $\beta = \frac{1}{2}$, consistent with the previous rescaling, then

$$0 = \alpha v + \frac{1}{2} y \cdot \nabla v + \Delta v.$$

Since Δ is rotationally symmetric (Homework 1) assume $v(y) = w(|y|)$, then

$$\begin{aligned} 0 &= \alpha v + \frac{1}{2} y \cdot \nabla v + \Delta v \\ &= \alpha w + \frac{1}{2} y \cdot \nabla |y| w' + w'' + \frac{n-1}{r} w' \\ &= \alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' \end{aligned}$$

Let $\alpha = \frac{n}{2}$, then

$$\begin{aligned} 0 &= \frac{n}{2}w + \frac{1}{2}rw' + w'' + \frac{n-1}{r}w' \\ &= \frac{1}{r^{n-1}}\left[(r^{n-1}w')' + \frac{1}{2}(r^n w)'\right] \end{aligned}$$

implying

$$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0$$

and therefore

$$r^{n-1}w' + \frac{1}{2}r^n w = c$$

We want w and w to vanish at $r \to \infty$, therefore $c = 0$ implying

$$w' = -\frac{1}{2}rw$$

which is solved by

$$v(y) = w(r) = \tilde{c}e^{-\frac{r^2}{4}} = \tilde{c}e^{-\frac{|y|^2}{4}}$$

and

$$u(x,t) = \frac{\tilde{c}}{t^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}.$$

Definition 33 (Fundamental Solution)
The function

$$\varphi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

is called the fundamental solution of the heat equation.

Remark 34

1. Note that there is a singularity in $(0,0)$.
2. The prefactor ensures $\int_{\mathbb{R}^n} \varphi(x,t)\,dx = 1$ for all $t > 0$.
3. By construction $\varphi_t - \Delta\varphi = 0$

Theorem 35 (Solution of the initial value problem)
Assume $g \in C_b^0(\mathbb{R}^n)$, then u defined by

$$u(x,t) = \varphi(x,t) * g(x) = \int_{\mathbb{R}^n} \varphi(x-y,t)g(y)\,dy$$

satisfies

a) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

b) $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$

c) $\lim\limits_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(x,t) = g(x_0)$ for all $x_0 \in \mathbb{R}^n$

Lecture 7 (January 28)

Proof

a) Let $\delta > 0$, then $\frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ is C^∞ on $\mathbb{R}^n \times [\delta,\infty)$ with uniformly bounded derivatives. Therefore we can swap integration and differentiation and find that $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$.

b) Since we can swap integration and differentiation and φ solves the heat equation and shifting by y leaves the equation unchanged (Homework 1)

$$u_t(x,t) - \Delta u(x,t) = \int_{\mathbb{R}^n} [(\varphi_t - \Delta_x \varphi)(x-y,t)] g(y)\, dy = 0$$

in $\mathbb{R}^n \times (0,\infty)$

c) Fix x_0, ε. Choose $\delta > 0$ such that

$$|y - x_0| < \delta \quad\implies\quad |g(y) - g(x_0)| < \varepsilon$$

If $|x - x_0| < \frac{\delta}{2}$, then

$$\begin{aligned}
|u(x,t) - g(x_0)| &= \left| \int_{\mathbb{R}^n} \varphi(x-y,t)\,[g(y) - g(x_0)]\ dy \right| \\
&\le \int_{B_\delta(x_0)} \varphi(x-y,t)\,|g(y) - g(x_0)|\ dy \\
&\qquad + \underbrace{\int_{\mathbb{R}^n \setminus B_\delta(x_0)} \varphi(x-y,t)\,|g(y) - g(x_0)|\ dy}_{J} \\
&\le \max_{y \in B_\delta(x_0)} |g(y) - g(x_0)| \underbrace{\int_{\mathbb{R}^n} \varphi(x-y,t)\, dy}_{=1} + J \\
&\le \varepsilon + J
\end{aligned}$$

And

$$\begin{aligned}
J &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n \setminus B_\delta(x_0)} \varphi(x-y,t)\, dy \\
&= Ct^{-\frac{n}{2}} \int_{\mathbb{R}^n \setminus B_\delta(x_0)} e^{-\frac{|x-y|^2}{4t}}\, dy \\
&\overset{\star}{\leq} Ct^{-\frac{n}{2}} \int_{\mathbb{R}^n \setminus B_\delta(x_0)} e^{-\frac{|x_0-y|^2}{16t}}\, dy \\
&\leq Ct^{-\frac{n}{2}} \int_\delta^\infty r^{n-1} e^{-\frac{r^2}{16t}}\, dr \\
&\overset{\rho=\frac{r}{\sqrt{t}}}{=} C \int_{\frac{\delta}{\sqrt{t}}}^\infty \rho^{n-1} e^{-\rho^2}\, d\rho \\
&\overset{t\to 0}{\to} 0
\end{aligned}$$

where in $\star$ we used

$$\begin{aligned}
&|y-x_0| \overset{|x-x_0|\leq\frac{\delta}{2}}{\leq} |y-x| + \frac{\delta}{2} \overset{\delta\leq|y-x_0|}{\leq} |y-x| + \frac{1}{2}|y-x_0| \\
\implies &|y-x_0| \leq 2|y-x|
\end{aligned}$$

Therefore for t sufficiently small

$$|u(x,t) - g(x_0)| \leq 2\varepsilon.$$

□

Remark 36 (Interpretation of the fundamental solution)
Since

$$\varphi_t - \Delta\varphi = 0$$

and

$$\int_{\mathbb{R}^n} \varphi(x-y,t) g(y) \overset{t\to 0}{\to} g(x)$$

formally the fundamental solution solves

$$\begin{aligned}
\varphi_t - \Delta\varphi &= 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty) \\
\varphi &= \delta_0 \qquad \text{on } \mathbb{R}^n \times \{t=0\}
\end{aligned}$$

Remark 37 (Infinite propagation speed)
Note that if $g \in C_b^0(\mathbb{R}^n)$ and $0 \not\equiv g \geq 0$, then

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy > 0$$

for all $x, t \in \mathbb{R}^n \times (0,\infty)$. So information of g is being propagated to any $x \in \mathbb{R}^n$ immediately.

Inhomogeneous problem

For the inhomogeneous problem

$$u_t - \Delta u = f$$

note that by shifting (Homework 1) for fixed s

$$u(x,t;s) = \int_{\mathbb{R}^n} \varphi(x-y, t-s) f(y,s)\ dy$$

solves

$$\begin{aligned} u_t(\cdot;s) - \Delta u(\cdot;s) &= 0 && \text{in } \mathbb{R}^n \times (s,\infty) \\ u_t(\cdot;s) &= f(\cdot, s) && \text{on } \mathbb{R}^n \times \{t=s\} \end{aligned}$$

Think about an initial value where the initial value is $f(\cdot,s)\,ds$ If we integrate in s over these initial values we find the solution. This is called Duhamel's principle.

Theorem 38
For $f \in C^2_{1,c}(\mathbb{R}^n \times [0,\infty))$ C^2 in space and C^1 in time + compactly supported and

$$\begin{aligned} u(x,t) &= \varphi *_{x,t} f \\ &= \int_0^t \int_{\mathbb{R}^n} \varphi(x-y, t-s)\, f(y,s)\, dy\, ds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy\, ds \end{aligned}$$

one has

1. $u, u_t, \nabla u, \nabla^2 u \in C^0(\mathbb{R}^n \times (0,\infty))$
2. $u_t - \Delta u = f$ in $\mathbb{R}^n \times (0,\infty)$
3. $\lim_{(x,t)\to(x_0,0)} u(x,t) = 0$

Proof

1. Similar to the Laplace equation at $t > 0$ the integrand is smooth so we can swap integration and differentiation and swap the integration variables such that

   $$\partial_t u(x,t) = \int_0^t \int_{\mathbb{R}^n} \varphi(y,s) f_t(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \varphi(y,t) f(x-y, 0)\, dy$$

   $$\in C^0$$

   and similar ∇, ∇^2.

2. Calculate

$$\begin{aligned}
u_t - \Delta u &= \int_0^t \int_{\mathbb{R}^n} \varphi(y,s)\left[(\partial_t - \Delta_x) f(x-y,t-s)\right] dy\, ds \\
&\quad + \int_{\mathbb{R}^n} \varphi(y,t) f(x-y,0)\, dy \\
&= \int_0^\varepsilon \int_{\mathbb{R}^n} \varphi(y,s)\left[(\partial_t - \Delta_x) f(x-y,t-s)\right] dy\, ds \\
&\quad + \int_\varepsilon^t \int_{\mathbb{R}^n} \varphi(y,s)\left[(-\partial_s - \Delta_y) f(x-y,t-s)\right] dy\, ds \\
&\quad + \int_{\mathbb{R}^n} \varphi(y,t) f(x-y,0)\, dy \\
&= I_\varepsilon + J_\varepsilon + K. \qquad (8.1)
\end{aligned}$$

One has

$$I_\varepsilon \le (\|\partial_t f\|_{L^\infty} + \|\nabla^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} \varphi(y,s)\, dy\, ds \le \varepsilon C \qquad (8.2)$$

and integrating by parts

$$\begin{aligned}
J_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} \underbrace{([\partial_s - \Delta_y)\varphi(y,s)]}_{=0} f(x-y,t-s)\, dy\, ds \\
&\quad + \int_{\mathbb{R}^n} \varphi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy \\
&\quad - \int_{\mathbb{R}^n} \varphi(y,t) f(x-y,0)\, dy \\
&= \int_{\mathbb{R}^n} \varphi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy - K \qquad (8.3)
\end{aligned}$$

Therefore

$$\begin{aligned}
u_t - \Delta u &\overset{(8.1),(8.3)}{=} \lim_{\varepsilon\to 0} \left(I_\varepsilon + \int_{\mathbb{R}^n} \varphi(y,\varepsilon) f(x-y,t-\varepsilon)\, dy \right) \\
&\overset{(8.2),\star}{=} f(x,t)
\end{aligned}$$

where in $\star$ one can do exactly the same calculations in Theorem 35 c).

□

Remark 39
Combining the two previous theorems

$$\begin{aligned}
u(x,t) &= \varphi *_{x,t} f + \varphi *_x g \\
&= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy\, ds \\
&\quad + \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy
\end{aligned}$$

solves

$$\begin{aligned} u_t - \Delta u &= f \qquad \text{in } \mathbb{R}^n \times (0,t) \\ u &\overset{\rightrightarrows}{=} g \qquad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

where $*_{x,t}$ means convolution in space and time and $*_x$ means convolution in space and $\overset{\rightrightarrows}{=}$ means in the limit $t \to 0$ as in in Theorem 35 c).

Lecture 8 (February 02)

8.2 Bounded domains

Here we will always assume U to be bounded

Definition 40
Fix $T > 0$.

1. The parabolic cylinder is
$$U_T = U \times (0,T]$$
2. The parabolic boundary is
$$\partial_p U_T = \overline{U_T} \setminus U_T$$

Remark 41
Note that the "top" $U \times \{t = T\}$ is included in the parabolic cylinder, and excluded from the parabolic boundary.

Definition 42
For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$ the heat ball is defined by

$$E_r(x,t) = \left\{ (y,s) \in \mathbb{R}^{n+1} \,\middle|\, s \le t,\ \varphi(x-y, t-s) \ge \frac{1}{r^n} \right\}.$$

Note that the heat ball $E_r(x,t)$ is in the past of (x,t).

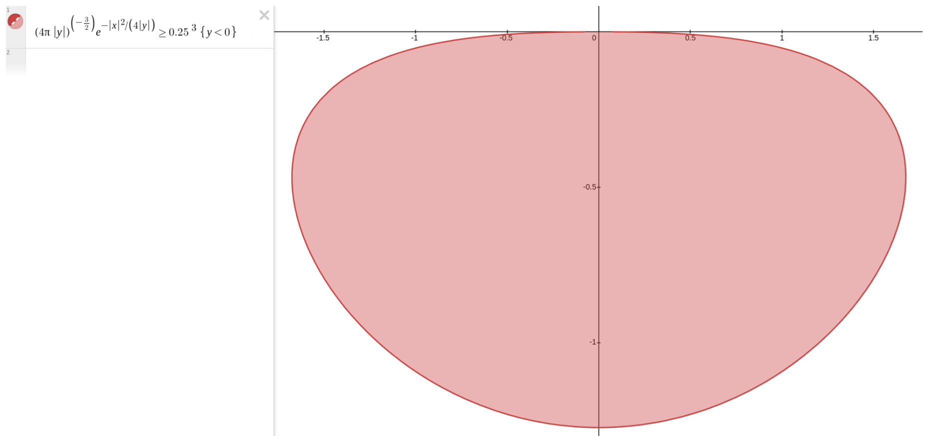

Figure B.3: Heat ball for $x = 0, t = 0, r = 0.25$ in $n = 3$ done in desmos

Mean value property

Theorem 43 (Mean value property)
Let $u \in C_1^2(U_T)$ solve

$$u_t - \Delta u = 0$$

then

$$u(x,t) = \frac{1}{4r^n} \iint_{E_r(x,t)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dyds$$

for each $E_r(x,t) \subset U_T$.

Proof
W.l.o.g assume $x = 0$, $t = 0$, u smooth (after potential mollification). Define

$$\rho(r) = \frac{1}{r^n} \iint_{\underbrace{E_r}_{=E_r(0,0)}} u(y,s) \frac{|y|^2}{s^2} \, dyds$$

$$(y,s) = (rz, r^2\tau) \rightarrow \quad = \iint_{E_1} u(rz, r^2\tau) \frac{|z|^2}{\tau^2} \, dz d\tau$$

Then

$$\rho'(r) = \iint_{E_1} (\nabla_y u_y(rz, r^2\tau) \cdot z + u_s(rz, r^2\tau) 2r\tau) \frac{|z|^2}{\tau^2} \, dz d\tau$$

$$= \underbrace{\frac{1}{r^{n+1}} \iint_{E_r} y \cdot \nabla_y u(y,s) \frac{|y|^2}{s^2} \, dyds}_{=A} + \underbrace{\frac{1}{r^{n+1}} \iint_{E_r} 2u_s(y,s) \frac{|y|^2}{s} \, dyds}_{B}$$

so

Introduce

$$\psi = \log(\varphi(y,-s)) + \log r^n = \log\left(\frac{1}{(-4\pi s)^{\frac{n}{2}}} e^{\frac{|y|^2}{4s}}\right) + \log r^n$$
$$= -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r,$$

then

$$\nabla_y \psi = \frac{1}{2s} y, \tag{8.4}$$
$$\psi_s = -\frac{n}{2s} - \frac{|y|^2}{4s^2}$$

and since $\partial E_r = \{(y,s)\,|\,s \leq 0,\ \varphi(y,-s) = r^{-n}\}$ we have

$$\psi|_{\partial E_r} = 0$$

by construction. Therefore

$$B = \frac{1}{r^{n+1}} \iint_{E_r} 2u_s(y,s)\frac{|z|^2}{s}\,dyds$$
$$= \frac{4}{r^{n+1}} \iint_{E_r} u_s y \cdot \nabla\psi\,dyds$$
$$\overset{\text{int by parts}}{=} -\frac{4}{r^{n+1}} \iint_{E_r} u_s n\psi\,dyds - \frac{4}{r^{n+1}} \iint_{E_r} \psi y \cdot \nabla u_s\,dyds$$

where the boundary integrals vanish since $\varphi = 0$ on ∂E_r. Now integrate by parts with respect to time

$$B = -\frac{4}{r^{n+1}} \iint_{E_r} u_s n\psi\,dyds + \frac{4}{r^{n+1}} \iint_{E_r} \psi_s y \cdot \nabla u\,dyds$$
$$= -\frac{4}{r^{n+1}} \iint_{E_r} u_s n\psi\,dyds - \frac{4}{r^{n+1}} \iint_{E_r} \frac{n}{2s} y \cdot \nabla u\,dyds$$
$$\underbrace{- \frac{4}{r^{n+1}} \iint_{E_r} \frac{|y|^2}{4s^2} y \cdot \nabla u\,dyds}_{=A}$$

So

$$\begin{aligned}
\rho'(r) &= -\frac{4}{r^{n+1}} \iint_{E_r} u_s n\psi \, dyds - \frac{4}{r^{n+1}} \iint_{E_r} \frac{n}{2s} y \cdot \nabla u \, dyds \\
&\overset{u_s - \Delta u = 0}{=} -\frac{4}{r^{n+1}} \iint_{E_r} \Delta u n\psi \, dyds - \frac{4}{r^{n+1}} \iint_{E_r} \frac{n}{2s} y \cdot \nabla u \, dyds \\
&\overset{pi}{=} \frac{4}{r^{n+1}} \iint_{E_r} n\nabla u \cdot \nabla\psi \, dyds - \frac{4}{r^{n+1}} \iint_{E_r} \frac{n}{2s} y \cdot \nabla u \, dyds \\
&= \frac{4}{r^{n+1}} \iint_{E_r} \left(\nabla\psi n - \frac{n}{2s} y \right) \cdot \nabla u \, dyds \\
&\overset{(8.4)}{=} 0
\end{aligned}$$

Therefore $\varphi(r)$ is constant and

$$\begin{aligned}
\rho(r) &= \lim_{r\to 0} \rho(r) = \lim_{r\to 0} \frac{1}{r^n} \iint_{E_r} u(y,s) \frac{|y|^2}{s^2} \, dyds \\
&\overset{\star}{=} 4u(0,0)
\end{aligned} \tag{8.5}$$

$\star$ can be seen by computing

$$\frac{1}{r^n} \iint_{E_r} \frac{|y|^2}{s^2} \, dyds = \iint_{E_1} \frac{|z|^2}{\tau^2} \, dzd\tau = \overset{\text{spherical coordinates}}{\ldots} = 4$$

so (8.5) is a weighted average integral with weight $\frac{|y|^2}{s^2}$. ☐

Remark 44

1. In the mean value property the right-hand side only depends on previous times, which is consistent with causality, i.e. it should not depend on the future.

2. It is a weighted average integral.

3. Similar to the laplace equation for $u(t,x) \leq$ "mean value" is a subsolution subsolutions and one can prove that for $u \in C_1^2(U_T)$

$$u(x,t) \leq \text{"mean value"} \quad \forall E_r(x,t) \subset U_T \qquad \Leftrightarrow \qquad u_t - \Delta u \leq 0.$$

4. Similar supersolutions

$$u(x,t) \geq \text{"mean value"} \quad \forall E_r(x,t) \subset U_T \qquad \Leftrightarrow \qquad u_t - \Delta u \geq 0.$$

   and strict inequality implies strict inequality

Maximum principle

Theorem 45
Assume $U \subset \mathbb{R}^n$ is open, $T \in (0, \infty]$ and $u \in C_1^2(U_T)$.

1. If $u_t - \Delta u < 0$ on U_T, then u has no local maximum in U_T.
2. If U is bounded, $u_t - \Delta u \leq 0$ on U_T, and $u \in C^0(U \cap \partial_p U_T)$, then

$$u \leq \sup_{\partial_p U_T} \qquad \text{in } U_T$$

   remember $\partial_p U_T$ is the parabolic boundary $\Omega \times \{t = 0\} \cup \partial U \times (0, T)$. (draw picture!)

3. Assume U is bounded and connected. If u solves $u_t - \Delta u \leq 0$ on U_T and reaches $M = \sup_{U_T} u < \infty$ at $(\tau, y) \in U_T$, then $u = M$ in all of $(0, \tau] \times U$.

Sketch of Proof

1. Homework
2. Homework
3.
   - First show there are line segments going back in time on which u is constant:
     Let $L_1 \subset U_T$ be a line segment from (y, τ) to $(t_1, x_1) \in U_T$. Let $(x, t) \in L_1 \cap \{u = M\} \ni (y, \tau)$ (so $L_1 \cap \{u = M\} \neq \varnothing$) be such that t is minimal (going further down the line $u < M$). Then for $0 < r \ll 1$, by the mean value property

$$\begin{aligned} M = u(x,t) &\leq \frac{1}{4r^n} \int_{E_r(x,t)} u(s,z) \frac{|z-x|^2}{(t-s)^2} \, dzds \\ &\leq M \frac{1}{4r^n} \int_{E_r(x,t)} \frac{|z-x|^2}{(t-s)^2} \, dzds = M \end{aligned}$$

     Therefore equality holds and by continuity $u(s, z) = M$ for all $(s, z) \in E_r(x, t)$. But $E_r(x, t)$ is in the past showing that $u = M$ for previous times, a contradiction to t being minimal, so $L_1 \cap \{u = M\} = L_1$ and $t = t_1$ and $u(x_1, t_1) = M$.
   - Since the slope of the boundary of the heat ball in (x, t) is 0, i.e. it is flat draw heat ball!, we can reach any (x_1, t_1) from (y, τ) if the line from x_1 to y is in U.
   - Any $(t_\star, x_\star) \in U_\tau$ can be reach by adding finitely many of those line segments, i.e. from $(y, \tau) \to (t_1, x_1) \to (t_2, x_2) \to \cdots \to (t_\star, x_\star)$, where $t_\star = t_k < \cdots < t_1 < \tau$ draw polygon!
   - So $u = M$ in $U \times (0, \tau)$ and by continuity in $U \times (0, \tau]$.

□

Remark 46
There are solutions such that $u = 0$ in U_τ, but $u(x,t) > 0$ for $t > \tau$. Interpretation: *switch on heating on the boundary.*

Lecture 9 (February 04)

Remark 47
Similar to before

1. Minimum principles

   Use maximum principle for $-u$.

2. Comparison principle

   U bounded,

$$\begin{aligned} u_t - \Delta u &\leq v_t - \Delta v && \text{in } U_T \\ u &\leq v && \text{on } \partial_p U_T \end{aligned}$$

   implies

$$u \leq v \qquad \text{in } U_T$$

3. Uniqueness of solutions

   If U is bounded, there exists at most one solution $u \in C_1^2(U_T) \cap C^0(U_T \cup \partial_p U_T)$ to the initial boundary value problem

$$\begin{aligned} u_t - \Delta u &= f && \text{in } U_T && (8.6) \\ u(\cdot, 0) &= u_0 && \text{on } U \\ u &= g && \text{on } (\partial U)_T && (8.7) \end{aligned}$$

   for given f, u_0, g.

4. sup estimates

   For solutions of (8.6)-(8.7)

$$\sup_{U_T} |u| \leq \max\left\{ \sup_U |u_0|, \sup_{(\partial U)_T} |g| \right\} + T \sup_{U_T} |f| \tag{8.8}$$

$$\sup_{U_T} |u| \leq \max\left\{ \sup_U |u_0|, \sup_{(\partial U)_T} |g| \right\} + \frac{R^2}{2n} \sup_{U_T} |f| \quad \text{if } U \subset B_R(x_0) \tag{8.9}$$

   (8.8) stricter if T is small, (8.9) stricter if U is small or $T = \infty$

   Sketch of Proof Apply max/min principle to $w(x,t) = u(x,t) \mp t \sup_{U_T} |f|$ and $w(x,t) = u(x,t) \pm \frac{|x-x_0|^2}{2n} \sup_{U_T} |f|$ □

5. Continuous dependence

   Apply 4 to $u_1 - u_2$.

6. For $U = \mathbb{R}^n$ one needs additional growth conditions.

   Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T])$ solves

$$\begin{aligned} u_t - \Delta u = 0 \qquad & \text{in } \mathbb{R}^n \times (0,T) \\ u(\cdot, 0) = g \qquad & \text{on } \mathbb{R}^n \end{aligned}$$

   and satisfies

$$u(x,t) \leq A e^{a(T)|x|^2}$$

   for all $(x,t) \in \mathbb{R}^n \times [0,T]$, then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$$

7. The growth conditions is crucial for the full $\mathbb{R}^n$.

   There are solutions to

$$\begin{aligned} u_t - \Delta u = 0 \qquad & \text{in } \mathbb{R}^n \times (0,T) \\ u(\cdot, 0) = 0 \qquad & \text{on } \mathbb{R}^n \end{aligned}$$

   with $u(x,t) \neq 0$ for $(x,t) \in U_T$. These solutions grow rapidely at $|x| \to \infty$.

   Sketch of Proof

   (a) φ solves $\varphi_t - \Delta \varphi = 0$ on $\mathbb{R}^{n+1} \setminus \{0,0\}$.

   (b) Power series in $1d$

   - $n = 1$, $U = \mathbb{R}$ and

$$u(x,t) = \sum_{k=0}^{\infty} h^{(k)}(t) \frac{x^{2k}}{(2k)!} \qquad h^{(k)} = k\text{th derivative of } h$$

     with $h \in C^\infty(\mathbb{R})$.

   - If $\frac{h^{(k)}(t)}{(2k)!} \overset{k \to \infty}{\to} 0$ locally uniform in time faster than exponential we can swap differentiation and sum. Then

$$\begin{aligned} \Delta u &= \frac{d^2}{dx^2} \left(\sum_{k=0} h^{(k)}(t) \frac{x^{2k}}{(2k)!} \right) = \sum_{k=1} h^{(k)} \frac{x^{2(k-1)}}{(2(k-1))!} \\ &= \sum_{k=0} h^{(k+1)} \frac{x^{2k}}{(2k)!} = u_t \end{aligned}$$

   - $h(t) = e^t$ gives $u(x,t) = e^t \cosh x$ which is fine

- but

$$h(t) = \begin{cases} e^{-\frac{1}{t^2}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

gives C^∞ solution on $(-\infty, \infty) \times \mathbb{R}$ and $u = 0$ on $(-\infty, 0] \times \mathbb{R}$ but $u \neq 0$ everywhere in $(0, \infty) \times \mathbb{R}$.

□

8.3 Further Concepts

Energy Methods

We can show many aspects via *energy methods* which is a much more general concept than what we have seen here. The idea is to introduce an *energy* and showing that this is bounded or dissipated. We will shortly reprove the uniqueness solutions.

Lemma 48

If U is bounded, there exists at most one solution $u \in C_1^2(U_T) \cap C^0(U_T \cup \partial_p U_T)$ to the initial boundary value problem

$$u_t - \Delta u = f \qquad \text{in } U_T \tag{8.10}$$
$$u(\cdot, 0) = u_0 \qquad \text{on } U$$
$$u = g \qquad \text{on } (\partial U)_T \tag{8.11}$$

for given f, u_0, g.

Proof

Assume u and $\tilde{u}$ solve (8.10)-(8.11), then $w = u - \tilde{u}$ solves

$$w_t - \Delta w = 0 \qquad \text{in } U_T$$
$$w(\cdot, 0) = 0 \qquad \text{on } U$$
$$w = 0 \qquad \text{on } (\partial U)_T$$

Then the energy

$$E(t) = \frac{1}{2} \int_U |w|^2(x,t)\, dx$$

satisfies

$$\frac{d}{dt} E(t) = \int_U w w_t \, dx = \int_U w \Delta w \, dx = \int_{\partial U} \underbrace{w}_{=0} n \cdot \nabla w \, dS(x) - \int_U \nabla w \cdot \nabla w \, dx$$

$$= - \int_U |\nabla \omega|^2 \, dx \leq 0$$

Therefore

$$\int_U |w|^2(x,t) = E(t) \leq E(0) = \int_U |w|^2(x,0)\ dx = 0$$

So $w = 0$ a.e. and by continuity $w = 0$. □

Fourier Methods

Consider U either a periodic domain (which is often done to remove any boundary effects) or the full space. If u is sufficiently smooth and integrable and solves

$$u_t - \Delta u = 0$$

then Fourier transforming in space

$$0 = \widehat{u_t} - \widehat{\Delta u} = \hat{u}_t(t,\xi) - (i\xi \cdot i\xi)\hat{u}(t,\xi) = \hat{u}_t(t,\xi) + |\xi|^2\hat{u}(t,\xi)$$

For any $\xi \in \mathbb{R}^n$ this is an ode in time

$$\hat{u}_t = -|\xi|^2\hat{u}(t,\xi)$$

solved by

$$\hat{u}(t,\xi) = \hat{u}(0,\xi)e^{-|\xi|^2 t}.$$

where $\hat{u}(0,\xi)$ is the Fourier transform of the initial data. So high wave numbers/frequencies (in space) are exponentially damped in time leading to smooth (and analytic) solutions.

In contrast the backwards heat equation

$$u_t + \Delta u = 0$$

is ill-posed and even analytic initial data can lead to singular solutions in finite time. Similar to before it heuristically solved by

$$\hat{u}(t,\xi) = \hat{u}(0,\xi)e^{|\xi|^2 t}.$$

Lecture 10 (February 09)

9 Wave Equation

The wave equation is given by

$$u_{tt} - \Delta u = 0$$

where again $u = u(x,t)$.

9.1 One dimension, Unbounded

In the homework we constructed the solution for the full space $n = 1$ dimensions.

Theorem 49
Assume $u_0 \in C^2(\mathbb{R})$, $v_0 \in C^1(\mathbb{R})$, then the following are equivalent

- d'Alambert's formula

$$u(x,t) = \frac{1}{2}\big(u_0(x+t) + u_0(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy$$

holds in $\mathbb{R} \times [0,\infty)$.

- $u \in C^2(\mathbb{R} \times [0,\infty))$ solves

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } \mathbb{R} \times (0,\infty) \\ u|_{t=0} &= u_0 && \text{on } \mathbb{R} \\ u_t|_{t=0} &= v_0 && \text{on } \mathbb{R} \end{aligned}$$

Remark 50

1. Proof in homework.
2. $u_0 \in C^k$, $v_0 \in C^{k-1}$ implies $u \in C^k$, so no regularity gain as in the heat equation or laplace equation.
3. speed of propagation is 1, for $u_{tt} - c^2 u_{xx} = 0$ the speed is c by rescaling.

Half Space

Consider $u_0 \in C^2(\mathbb{R}_+), v_0 \in C^1(\mathbb{R}_+)$ with $u_0(0) = v_0(0) = 0$ and u solves

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } \mathbb{R}_+ \times (0,\infty) \\ u|_{t=0} &= u_0 && \text{on } \mathbb{R}_+ \\ u_t|_{t=0} &= v_0 && \text{on } \mathbb{R}_+ \\ u &= 0 && \text{on } \{x=0\} \times (0,\infty) \end{aligned}$$

Then we can reflect/define

$$\begin{aligned} \tilde{u}(x,t) &= \begin{cases} u(x,t) & x \geq 0 \\ -u(-x,t) & x \leq 0 \end{cases} \\ \tilde{u}_0(x) &= \begin{cases} u_0(x) & x \geq 0 \\ -u_0(-x) & x \leq 0 \end{cases} \\ \tilde{v}_0(x) &= \begin{cases} v_0(x) & x \geq 0 \\ -v_0(-x) & x \leq 0 \end{cases} \end{aligned} \tag{9.1}$$

which solves

$$\begin{aligned} \tilde{u}_{tt} - \tilde{u}_{xx} &= 0 && \text{in } \mathbb{R} \setminus \{0\} \times (0,\infty) \\ \tilde{u}|_{t=0} &= \tilde{u}_0 && \text{on } \mathbb{R} \\ \tilde{u}_t|_{t=0} &= \tilde{v}_0 && \text{on } \mathbb{R} \end{aligned}$$

Note that we choose it so that the slopes match. $\tilde{u}$ actually solves $\tilde{u}_{tt} - \tilde{u}_{xx} = 0$ everywhere in a weak sense. Applying d'Alambert's formula

$$\tilde{u}(x,t) = \frac{1}{2}\big(\tilde{u}_0(x+t) + \tilde{u}_0(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{v}_0(y)\,dy$$

which for u implies

$$u(x,t) = \begin{cases} \frac{1}{2}\big(u_0(x+t) + u_0(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy & \text{if } 0 \le t \le x \\ \frac{1}{2}\big(u_0(x+t) - u_0(t-x)\big) + \frac{1}{2}\int_{t-x}^{x+t} v_0(y)\,dy & \text{if } 0 \le x \le t \end{cases} \tag{9.2}$$

Note that $u \in C^1$ since $u_0(0) = v_0(0) = 0$ and the slopes match, but In order to have $u \in C^2$ and to have a classical solution we additionally need $u_0''(0) = 0$ so that (9.1) is actually a second order approximation

9.2 Multiple dimensions

We construct the solutions by spherical means. The construction is quite lengthy and technical so just outline the ideas.

Let $m, n \ge 2$ and $u \in C^m(\mathbb{R}^n \times [0,\infty))$ solve

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} &= u_0 && \text{on } \mathbb{R}^n \\ u_t|_{t=0} &= v_0 && \text{on } \mathbb{R}^n \end{aligned}$$

and introduce

$$\begin{aligned} \bar{u}(x,r,t) &= \fint_{\partial B_r(x)} u(y,t)\,dS(y), \\ \bar{u}_0(x,r) &= \fint_{\partial B_r(x)} u_0(y)\,dS(y), \\ \bar{v}_0(x,r) &= \fint_{\partial B_r(x)} v_0(y)\,dS(y). \end{aligned}$$

Lemma 51 (Euler-Poisson-Darboux equation)
Then $\bar{u} \in C^m(\mathbb{R}_+ \times [0,\infty))$ solves

$$\begin{aligned} \bar{u}_{tt} - \bar{u}_{rr} - \frac{n-1}{r}\bar{u}_r &= 0 && \text{in } \mathbb{R}_+ \times (0,\infty) \\ \bar{u}|_{t=0} &= \bar{u}_0 && \text{on } \mathbb{R}_+ \\ \bar{u}_t|_{t=0} &= \bar{v}_0 && \text{on } \mathbb{R}_+ \end{aligned}$$

Remark 52

1. Remember $\partial_r^2 + \frac{n-1}{r}\partial_r$ is the radial part of Δ
2. We switched from an $n+1$ dimensional problem to a slightly more complicated 2 dimensional problem

Proof
As for the mean value formula of the laplacian, (7.6),

$$\bar{u}_r(x,r,t) = \underbrace{\frac{1}{n\omega_n r^{n-1}}}_{=\frac{1}{|\partial B_r|}} \int_0^r \int_{\partial B_\rho(x)} \Delta u(y,t)\, dS(y)\, d\rho$$

Then by product rule

$$\begin{aligned}
\bar{u}_{rr}(x,r,t) &= -\frac{n-1}{n\omega_n r^n} \int_{\partial B_r(x)} \Delta u(y,t)\, dy + \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} \Delta u(y,t)\, dS(y) \\
&= -\frac{n-1}{r} \bar{u}_r(x,r,t) + \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(0)} \Delta u(y+x,t)\, dS(y) \\
&= -\frac{n-1}{r} \bar{u}_r(x,r,t) + \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} u_{tt}(y,t)\, dS(y) \\
&= -\frac{n-1}{r} \bar{u}_r(x,r,t) + \bar{u}_{tt}(x,r,t)
\end{aligned}$$

□

n=3

Rescale

$$\begin{aligned}
\tilde{u}(x,r,t) &= r\bar{u}(x,r,t) \\
\tilde{u}(x,r) &= r\bar{u}(x,r) \\
\tilde{u}(x,r) &= r\bar{u}(x,r)
\end{aligned}$$

then

$$\begin{aligned}
\tilde{u}_{tt} - \tilde{u}_{rr} &= 0 && \text{in } \mathbb{R}_+ \times (0,\infty) \\
\tilde{u}|_{t=0} &= \tilde{u}_0 && \text{on } \mathbb{R}_+ \\
\tilde{u}_t|_{t=0} &= \tilde{v}_0 && \text{on } \mathbb{R}_+ \\
\tilde{u} &= 0 && \text{on } \{r=0\} \times (0,\infty)
\end{aligned}$$

since

$$\partial_r^2 \tilde{u} = \partial_r(\bar{u} + r\bar{u}_r) = 2\bar{u}_r + r\bar{u}_{rr} = r\left(\frac{n-1}{r}\bar{u}_r + \bar{u}_{rr}\right) = r\bar{u}_{tt} = \tilde{u}_{tt}$$

which is the half-space problem we solved earlier. Therefore, for $0 \le r \le t$ by (9.2)

$$\tilde{u}(x,r,t) = \frac{1}{2}\big(\tilde{u}_0(x,r+t) - \tilde{u}_0(x,t-r)\big) + \frac{1}{2}\int_{t-r}^{r+t} \tilde{v}_0(x,y)\, dy$$

What does this mean for our original solution u? Note that as the spherical average

$$
\begin{aligned}
u(x,t) &= \lim_{r\to 0} ⨍_{\partial B_r(x)} u(y,t)\,dS(y) = \lim_{r\to 0} \bar{u}(x,r,t) = \lim_{r\to 0} \frac{\tilde{u}(x,r,t)}{r} = \\
&= \partial_t \tilde{u}_0(x,t) + \tilde{v}_0(x,t) \\
&= \partial_t \left(t ⨍_{\partial B_t(x)} u_0(y)\ dS(y) \right) + t ⨍_{\partial B_t(x)} v_0(y)\ dS(y) \qquad (9.3) \\
z = \frac{y-x}{t} \to &= ⨍_{\partial B_t(x)} u_0(y) + t v_0(y)\ dS(y) + t\, \underbrace{\partial_t ⨍_{\partial B_1(0)} u_0(x+tz)\ dS(z)}_{= ⨍ z\cdot \nabla u_0(x+tz)} \\
&= ⨍_{\partial B_t(x)} u_0(y) + t v_0(y)\ dS(y) + t ⨍_{\partial B_t(x)} \frac{y-x}{t} \cdot \nabla u_0(y)\ dS(y) \\
&= ⨍_{\partial B_t(x)} u_0(y) + t v_0(y) + (y-x)\cdot \nabla u_0(y)\ dS(y) \qquad (9.4)
\end{aligned}
$$

Corollary 53 (Kirchhoff's formula)
(9.4) solves

$$
\begin{aligned}
u_{tt} - \Delta u &= 0 && \text{in } \mathbb{R}^3 \times (0,\infty) \\
u|_{t=0} &= u_0 && \text{on } \mathbb{R}^3 \\
u_t|_{t=0} &= v_0 && \text{on } \mathbb{R}^3
\end{aligned}
$$

Lecture 11 (February 11)

n=2

Theorem 54 (Poisson's formula)

$$
u(x,t) = \frac{1}{2} ⨍_{B(x,t)} \frac{t u_0(y) + t^2 v_0(y) + t(y-x)\cdot \nabla u_0(y)}{\sqrt{t^2 - |y-x|^2}}\, dy
$$

solves

$$
\begin{aligned}
u_{tt} - \Delta u &= 0 && \text{in } \mathbb{R}^2 \times (0,\infty) \\
u|_{t=0} &= u_0 && \text{on } \mathbb{R}^2 \\
u_t|_{t=0} &= v_0 && \text{on } \mathbb{R}^2
\end{aligned}
$$

The Euler-Poisson Darboux equations can not be solved the same way. Instead we just regard the 2d problem in 3d, called the method of descent.

Sketch of Proof
Define

$$\hat{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t),$$
$$\hat{u}_0(x_1, x_2, x_3, t) = u_0(x_1, x_2, t),$$
$$\hat{v}_0(x_1, x_2, x_3, t) = v_0(x_1, x_2, t),$$

which are just constant in x_3. Then

$$\begin{aligned} \hat{u}_{tt} - \Delta \hat{u} &= 0 && \text{in } \mathbb{R}^3 \times (0, \infty) \\ \hat{u}|_{t=0} &= \hat{u}_0 && \text{on } \mathbb{R}^3 \\ \hat{u}_t|_{t=0} &= \hat{v}_0 && \text{on } \mathbb{R}^3 \end{aligned}$$

and therefore with $x \in \mathbb{R}^2$ and $\hat{x} = (x_1, x_2, 0) \in \mathbb{R}^3$

$$\begin{aligned} u(x,t) &= \hat{u}(\hat{x}, t) \\ &\overset{(9.3)}{=} \partial_t \left(t \fint_{\partial \hat{B}_t(\hat{x})} \hat{u}_0(y)\ d\hat{S}(y) \right) + t \fint_{\partial \hat{B}_t(\hat{x})} \hat{v}_0(y)\ d\hat{S}(y) \end{aligned}$$

where $\hat{B}_t(\hat{x})$ is the $\mathbb{R}^3$ ball centered around $\hat{x}$ and $d\hat{S}$ the corresponding measure of $\partial \hat{B}_t(\hat{x})$. Note that

$$\begin{aligned} t \fint_{\partial \hat{B}_t(\hat{x})} \hat{v}_0(y)\ d\hat{S}(y) &= t \frac{1}{4\pi t^2} \int_{\partial \hat{B}_t(\hat{x})} \hat{v}_0(y)\ d\hat{S}(y) \\ &= t \frac{2}{4\pi t^2} \int_{B_t(x)} v_0(y) \sqrt{1 + |\nabla \gamma(y)|^2}\ dy, \\ &= \frac{t}{2} \fint_{B_t(x)} v_0(y) \sqrt{1 + |\nabla \gamma(y)|^2}\ dy, \end{aligned}$$

where $\gamma(y) = \sqrt{t^2 - |y-x|^2}$ is the parametrization of the sphere. 2 because of 2 hemispheres. Now

$$\sqrt{1 + |\nabla \gamma(y)|^2} = \sqrt{1 + \left| \frac{y-x}{\gamma(y)} \right|^2} = \frac{t}{\sqrt{t^2 - |y-x|^2}}.$$

Combining this yields

$$u(x,t) = \hat{u}(\hat{x}, t) = \cdots + \frac{1}{2} \fint_{B_t(x)} \frac{t^2 v_0(y)}{\sqrt{t^2 - |y-x|^2}}\ dy.$$

The other term works in the same fashion. □

odd dimensions

Theorem 55
Assume $n \geq 3$ is odd, $u_0 \in C^{\frac{n+3}{2}}(\mathbb{R}^n)$, $v_0 \in C^{\frac{n+1}{2}}(\mathbb{R}^n)$, then

$$\begin{aligned} u(x,t) = \frac{n}{n!!}\Bigg[&\partial_t \left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}} \left(t^{n-2} \fint_{\partial B_t(x)} u_0 \, dS\right) \\ &+ \left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}} \left(t^{n-2} \fint_{\partial B_t(x)} v_0 \, dS\right)\Bigg] \\ \in C^2(&\mathbb{R}^n \times [0,\infty)) \end{aligned}$$

where $n!! = n \cdot (n-2) \ldots 3 \cdot 1$ is C^2 and solves

$$\begin{aligned} u_{tt} - \Delta u &= 0 \qquad && \text{in } \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} &= u_0 && \text{on } \mathbb{R}^n \\ u_t|_{t=0} &= v_0 && \text{on } \mathbb{R}^n \end{aligned}$$

even dimensions

Theorem 56
Assume $n \geq 2$ is even, $u_0 \in C^{\frac{n+4}{2}}(\mathbb{R}^n)$, $v_0 \in C^{\frac{n+2}{2}}(\mathbb{R}^n)$, then

$$\begin{aligned} u(x,t) = \frac{1}{n!!}\Bigg[&\partial_t \left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}} \left(t^{n} \fint_{B_t(x)} \frac{u_0}{\sqrt{t^2 - |y-x|^2}} \, dy\right) \\ &+ \left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}} \left(t^{n} \fint_{B_t(x)} \frac{v_0}{\sqrt{t^2 - |y-x|^2}} \, dy\right)\Bigg] \\ \in C^2(&\mathbb{R}^n \times [0,\infty)) \end{aligned}$$

where $n!! = n \cdot (n-2) \ldots 4 \cdot 2$ and solves

$$\begin{aligned} u_{tt} - \Delta u &= 0 \qquad && \text{in } \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} &= u_0 && \text{on } \mathbb{R}^n \\ u_t|_{t=0} &= v_0 && \text{on } \mathbb{R}^n \end{aligned}$$

Huygens' principle

For even dimension $n \geq 2$, the solution is an integral over the Ball $B_t(x)$, so data $u_0(x_0)$ will affect u in some point (x_1, t) for all sufficiently large times.

An infinitely stretched drum membrane (modeled by the wave equation) in $2d$ will oscillate for all times.

In contrast in odd dimensions $n \geq 3$, the solution is an integral over the boundary of the ball $\partial B_t(x)$, so that $u_0(x_0)$ will only affect the solution u in (x_1, t) at one moment of time. So there is a wavefront.

An flash of light or a sonic boom in $3d$ propagates only on a shell/wave front.

Finite speed of propagation

The formula show that the speed of propagation is 1 and a point is only influenced by the backward wave cone and only influences the forwards wave cone. draw picture

This is strongly related to the light cone in general relativity.

9.3 Bounded Domains

As we saw in the $1d$ half space problem, solutions of the wave equation tend to be reflected on the boundary. This makes it tedious to study explicitly. What we can do is energy methods.

Theorem 57 (Uniqueness)
Let $U \subset \mathbb{R}^n$ be bounded with smooth boundary. There exists at most one solution $u \in C^2(U_T) \cap C^0(\overline{U_T})$ of

$$\begin{aligned} u_{tt} - \Delta u &= f && \text{in } U \times (0,T) \\ u &= g && \text{on } \partial U \times (0,T) \\ u|_{t=0} &= u_0 && \text{on } U \\ u_t|_{t=0} &= v_0 && \text{on } U \end{aligned}$$

Proof
In homework. □

Chapter C

Sobolev Spaces

The classical notion of a solution is too restrictive (e.g. sub/super solutions using mean value properties, fundamental solution of heat/laplace equation, the wave equation on the half space, ...). It is sometimes not even possible to find solutions to PDE since they themselves develop points of non-differentiability.

Why weak derivatives?

1. PDEs can be explained even for irregular data
   (a) $\Delta u = f$ can make sense even if f has jump discontinuities.
   (b) More importantly the have much better functional analytic properties which we will exploit to prove existence of solutions.

Idea
Integration by parts yields

$$\int_U u \frac{\partial \varphi}{\partial x_i} \, dx = - \int_U \frac{\partial u}{\partial x_i} \varphi \, dx$$

for $U \in \mathbb{R}^n$ open, $u, \varphi \in C^1(U)$ if either u or φ has compact support.

The left-hand side makes sense even if $u \in L^1$, $\varphi \in C^1$ and the right hand side if $\frac{\partial u}{\partial x_i} \in L^1$, $\varphi \in C^1$.

10 Weak Derivatives

10.1 Integrable functions as derivatives

Definition 58 (weak derivatives)
Consider $U \subset \mathbb{R}^n$ is open, $i \in \{1, 2, \ldots, n\}$, $\alpha \in \mathbb{N}_0^n$

- For $u, v \in L^1_{\text{loc}}(U)$, v is the weak i-th partial derivative of u on U, written "$v = \partial_i u$ weak on U" if it holds

$$\int_U u \underbrace{\frac{\partial \varphi}{\partial x_i}}_{\text{classical derivative}} dx = -\int v\varphi \, dx$$

  for all $\varphi \in \mathcal{D}(U) = C_c^\infty(U)$, the space of test functions.

- Similarly $v = \partial^\alpha u$ is the weak α-th partial derivative of u on U if

$$\int_U u \frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha} \, dx = (-1)^{|\alpha|} \int_U v\varphi \, dx$$

  for all $\varphi \in \mathcal{D}$.

- For vector valued functions $u, v \in L^1(U; \mathbb{R}^N)$ the definition is component wise

$$\partial^\alpha u = v \qquad \Leftrightarrow \qquad \partial^\alpha u^k = v^k \quad \forall k \in \{1, \dots, n\}.$$

Example 59
In $n = 1$ dimensions $u(x) = |x|$ has the weak derivative $v(x) = \text{sign}(x)$ on all of $\mathbb{R}$.

Assume $\text{supp}\varphi \subset (-M, M)$, then

$$\begin{aligned}
\int_{\mathbb{R}} |x|\varphi'(x) \, dx &= -\int_{-M}^0 x\varphi'(x) \, dx + \int_0^M x\varphi'(x) \, dx \\
&= -x\varphi(x)|_{x=0} + \int_{-M}^0 x'\varphi(x) \, dx \\
&\quad - x\varphi(x)|_{x=0} - \int_0^M x'\varphi(x) \, dx \\
&= -\left(\int_{-M}^0 (-1)\varphi(x) \, dx + \int_0^M 1\varphi(x) \, dx \right) \\
&= -\int_{\mathbb{R}} \text{sign}(x) \, \varphi(x) \, dx
\end{aligned}$$

Lecture 12 (February 23)

Remark 60

1. Uniqueness

   A weak derivative, if it exists is uniquely defined up to a set of measure zero.

Proof
Assume $v, \tilde{v} \in L^1_{\text{loc}}(U)$ satisfy

$$\int_U u \frac{\partial^{|\alpha|}}{\partial x^\alpha} \varphi \, dx = (-1)^{|\alpha|} \int v\varphi \, dx = (-1)^{|\alpha|} \int \tilde{v}\varphi \, dx$$

for all $\varphi \in \mathcal{D}$. Then

$$\int (v - \tilde{v})\varphi \, dx = 0$$

for all $\varphi \in \mathcal{D}$, hence $v - \tilde{v} = 0$ a.e. □

2. Linearity

   For $\partial^\alpha u = v$, $\partial^\alpha \tilde{u} = \tilde{v}$ one has $\partial^\alpha(u + r\tilde{u}) = v + r\tilde{v}$ for all $r \in \mathbb{R}$.

3. Classical derivatives are weak derivatives

   If $\frac{\partial u}{\partial x_i}$ exists for all $x \in U$, then $\partial_i u = \frac{\partial u}{\partial x_i}$. By construction and uniqueness.

4. For $n \geq 2$ one can even have isolated singularities.

   If $\frac{\partial u}{\partial x_i}$ exists only on $U \setminus \{a\}$ and $u, \frac{\partial u}{\partial x_i} \in C^0(U \setminus \{a\})$ and $u, \frac{\partial u}{\partial x_i} \in L^1_{\text{loc}}$, then u is weakly differentiable on <u>all</u> of U, with $\partial_i u = \frac{\partial u}{\partial x_i}$. Proof by Fubini and then integration by parts in x_i.

   This is the case for $u(x) = |x|^s$ with $s > 1 - n$. For $1 - n < s < 0$, this is unbounded at 0 but weakly differentiable on all of $\mathbb{R}^n$.

5. Even weaker if

   - - A is relatively closed in U (that means closed in the topology of U. For example if $U = (0, 1)$, then $A = (0, \frac{1}{2}]$ is relatively closed in U.) with $\mathcal{H}^{n-1}(A) = 0$ or
     - $\mathcal{H}^{n-1}(p(A)) = 0$ where $p(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ so along the x_i direction the set looks like it has measure 0

     and $\frac{\partial u}{\partial x_i}$ exists on $U \setminus A$ with $u, \frac{\partial u}{\partial x_i} \in C^0(U \setminus A) \cap L^1_{\text{loc}}(U)$ or
   - A is an $(n-1)$ dimensional submanifold of U and $\frac{\partial u}{\partial x_i}$ exists on $U \setminus A$ with $\frac{\partial u}{\partial x_i} \in C^0(U \setminus A) \cap L^1_{\text{loc}}(U)$ and $u \in C^0(U)$

   then the weak derivative exists on U, and $\partial_i u = \frac{\partial u}{\partial x_i}$ on U.

6. If the singularity is even larger, the weak derivative does not exist as a L^1_{loc} function. For example $u = \mathbb{1}_G$ with C^1 domain $G \Subset U$ or $u(x) = |x|^s$ with $s \leq 1 - n$.

10.2 Measures as derivatives

Remark/Definition 61 (Measures)

- A non-negative Radon measure μ on U is a locally finite measure on the Borel-σ-algebra $B(U)$ of U, i.e. a σ-additive mapping $\mu : B(U) \to [0, \infty]$ with $\mu(K) < \infty$ for all compact $K \subset U$. Radon measures satisfy

  $$\begin{aligned}\mu(A) &= \sup\{\mu(K) \,|\, K \subset A, K \text{ is compact in } U\} \\ &= \inf\{\mu(O) \,|\, A \subset O, O \text{ is open in } U\}\end{aligned}$$

  for all $A \in B(U)$.

- A signed Radon measure μ on U is the difference of non-negative Radon measures μ_1, μ_2 on U. $\mu(A) = \mu_1(A) - \mu_2(A) \in [-\infty, \infty]$ whenever $\mu_1(A)$ or $\mu_2(A)$ are not both ∞. Therefore at least for all $A \in B(U)$ with $A \Subset U$.

  One identifies $\mu_1 - \mu_2$ with $\tilde{\mu}_1 - \tilde{\mu}_2$ if $\mu_1(A) - \mu_2(A) = \tilde{\mu}_1(A) - \tilde{\mu}_2(A)$ for all $A \in B(U)$ with $A \Subset U$.

  The integral $\int_A f \, d\mu = \int_A f \, d\mu_1 - \int_A f \, d\mu_2 \in \mathbb{R}$ is at least defined for measurable $A \subset U$ and $f \in L^1(A; \mu_1 + \mu_2)$. The space of signed Radon measures on U is $\mathrm{RM}_{\mathrm{loc}}(U)$.

  $\mu = \mu_1 - \mu_2$ is not unique but there exists the Jordan decomposition $\mu = \mu_+ - \mu_-$ into non-negative Radon measures μ_+, μ_- on U that satisfy there exist Borel set $N, P \subset U$ with $U = P \cup N$, $P \cap N = \varnothing$ and $\mu = \mu_+$ on P and $\mu = -\mu_-$ on N.

Definition 62 (Measures as derivatives)
Let $U \subset \mathbb{R}^n$ be open, $\alpha \in \mathbb{N}_0^n$. A signed Radon measure μ on U is the weak α-th partial derivative or α-th measure derivative of $u \in L^1_{\mathrm{loc}}(U)$ on U, written "$\partial^\alpha u = \mu$ weak on U" if

$$\int_U u \frac{\partial^{|\alpha|} u}{\partial x^\alpha} \, dx = (-1)^{|\alpha|} \int_U \varphi \, d\mu$$

holds for all $\varphi \in C_c^\infty(U)$.

The space of all $u \in L^1_{\mathrm{loc}}(U)$ such that $\partial^\alpha u$ for $|\alpha| \leq m$ exists weakly as signed Radon measures is denoted by $\mathrm{BV}^m_{\mathrm{loc}}(U)$. Specifically, $\mathrm{BV}_{\mathrm{loc}}(U) = \mathrm{BV}^1_{\mathrm{loc}}(U)$ is known as the space of functions of locally bounded variation.

Remark 63

1. Weighted Lebesgue measures are signed Radon measures, i.e. $v\mathcal{L}^n \in \mathrm{RM}_{\mathrm{loc}}(U)$ with weight $v \in L^1_{\mathrm{loc}}(U)$. They are given by

   $$v\mathcal{L}^n(A) = \int_A v(x) \, dx$$

   for all $A \in B(U)$ with $A \Subset U$.

2. One has the following characterization

$$v = \partial^\alpha u \text{ exists weakly in } L^1_{\text{loc}}(U)$$
$$\Leftrightarrow$$
$$\mu = \partial^\alpha u \text{ exists weakly in } \mathrm{RM}_{\text{loc}}(U) \text{ and is a weighted Lebesgue measure}$$

   and then $\mu = v\mathcal{L}^n$

3. Already for $n = 1$ dimensions derivative measures are more general than weighted Lebesgue measures. For example $\partial_x \mathrm{sign}(x) = 2\delta_0$.

4. Again they are unique (in the equivalence class of signed Radon measures) and linear

Even measures are not enough!

10.3 Distributions as derivatives

Definition 64 (Distribution)
A distribution (or generalized function) T on open $U \subset \mathbb{R}^n$ is a continuous linear functional

$$T : C_c^\infty(U) \to \mathbb{R}, \qquad \varphi \mapsto= \langle T, \varphi\rangle (= T(\varphi))$$

on the space of test function $C_c^\infty(U)$.

$\mathcal{D}(U) := C_c^\infty(U)$ and the space of distributions on U is denoted by $\mathcal{D}^\star(U)$ or $(C_c^\infty)^\star(U)$.

For every distribution $T \in \mathcal{D}^\star(U)$ one defines the distributional derivative $\partial^\alpha T \in \mathcal{D}^\star(U)$ by

$$\langle \partial^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\left\langle T, \frac{\partial^{|\alpha|}}{\partial x^\alpha}\varphi\right\rangle$$

Remark 65

1. Continuity is to be understood in the following rather weak sense

$$\left\{\begin{matrix} \varphi_k \text{ sequence in } C_c^\infty, \varphi \in C_c^\infty \\ \forall \alpha \in \mathbb{N}_0^n : \partial^\alpha \varphi_k \overset{k\to\infty}{\to} \partial^\alpha\varphi \text{ uniformly in } U \\ \bigcup_{k=1}^\infty \mathrm{supp}\,\varphi_k \Subset U \end{matrix}\right\} \implies T(\varphi_k) \overset{k\to\infty}{\to} T(\varphi)$$

2. $f \in L^1_{\text{loc}}(U)$ is identified with $T_f \in \mathcal{D}^\star(U)$ given by

$$\langle T_f, \varphi\rangle = \int_U f\varphi\, dx$$

   for all $\varphi \in \mathcal{D}(U)$.

3. $\mu \in \mathrm{RM}_{\mathrm{loc}}(U)$ is identified with $T_\mu \in \mathcal{D}^\star(U)$ given by

$$\langle T_\mu, \varphi \rangle = \int_U \varphi \, d\mu$$

for all $\varphi \in \mathcal{D}(U)$.

4. Every L^1_{loc} function, every signed Radon measure, and every distribution may be differentiated and their derivatives always exist (at least) as a distribution.

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5. For every distribution

$$\partial^\alpha(\partial^\beta T) = \partial^{\alpha+\beta} T = \partial^\beta(\partial^\alpha T)$$

since $\partial^\alpha \partial^\beta \varphi = \partial^\beta \partial^\alpha \varphi$.

Similarly $\partial^\alpha \partial^\beta f = \partial^\beta \partial^\alpha f$ for $f \in L^1_{\mathrm{loc}}$ and $\partial^\alpha \partial^\beta \mu = \partial^\beta \partial^\alpha \mu$ for $f \in \mathrm{RM}_{\mathrm{loc}}$ if these derivatives exist.

Note that $\partial^\alpha \partial^\beta f \in L^1_{\mathrm{loc}}/\mathrm{RM}_{\mathrm{loc}}$ does not imply $\partial^\beta f \in L^1_{\mathrm{loc}}/\mathrm{RM}_{\mathrm{loc}}$. This can be seen in by $u(x_1, x_2) = f(x_1) + g(x_2)$, since then $\partial_1 \partial_2 u = 0$ but f, g might not be weakly differentiable in L^1.

11 Properties of Derivatives

Theorem 66 (Mollification commutes with differentiation)
For $U \subset \mathbb{R}^n$ open, $u \in L^1_{\mathrm{loc}}(U)$, $\alpha \in \mathbb{N}_0^n$. If $\partial^\alpha u$ exists weakly, then

$$(\partial^\alpha u)_\varepsilon = \frac{\partial^{|\alpha|}}{\partial x^\alpha}(u_\varepsilon)$$

in U_ε.

Proof

$$\begin{aligned} (\partial^\alpha u)_\varepsilon(x) &= \int_U \partial^\alpha u(y) \eta_\varepsilon(x-y)\, dy \\ \text{def weak der} \to \quad &= (-1)^{|\alpha|} \int_U u(y) \frac{\partial^{|\alpha|}}{\partial y^\alpha} \eta_\varepsilon(x-y)\, dy \\ &= \int_U u(y) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \eta_\varepsilon(x-y)\, dy \\ &= \frac{\partial^{|\alpha|}}{\partial x^\alpha} \int_U u(y) \eta_\varepsilon(x-y)\, dy \end{aligned}$$

□

Works similarly for measures and distributions with $\mu_\varepsilon(x) = \eta_\varepsilon * \mu(x) = \int_U \eta_\varepsilon(x-y)\,d\mu(y)$, $T_\varepsilon(x) = \langle T, \eta_\varepsilon(x-\cdot)\rangle$, but a bit justification for the step for distributions needed.

Corollary 67 (Constancy)
If $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and for all $i = 1,\dots,n$ $\partial_i u = 0$ on $U \subset \mathbb{R}^n$, then u is constant in U.

Proof
Let $\varepsilon > 0$. Since $0 = \partial_i u$ we have $0 = (\partial_i u)_\varepsilon$ and by the previous theorem $0 = (\partial_i u)_\varepsilon = \frac{\partial}{\partial x_i}(u_\varepsilon)$. Therefore u_ε is locally constant on U_ε and since $u_\varepsilon \to u$ a.e. (Homework Q1.4) u is constant a.e. in U_ε. □

Theorem 68 (Equivalence of weak and strong derivatives)
For $U \subset \mathbb{R}^n$ open, $u, v \in L^1_{\text{loc}}(U)$, $\alpha \in \mathbb{N}_0^n$

$$\partial^\alpha u = v \text{ weak} \Leftrightarrow \left\{ \begin{matrix} \exists \text{ sequence } u_l \in C^\infty(U) \text{ with} \\ u_l \overset{l\to\infty}{\to} u,\ \partial^\alpha u_l \overset{l\to\infty}{\to} v \text{ in } L^1_{\text{loc}}(U) \end{matrix} \right\}$$

(Convergence in $L^1_{\text{loc}}(U)$ means $\lim_{l\to\infty} \|u_l - u\|_{L^1(K)} = 0$ for all $K \Subset U$.)

Proof

" $\Longrightarrow$ " Mollification: Choose compact $K_1 \subset K_2 \subset \dots$ with $\cup_{l=1}^\infty K_l = U$, $\varepsilon_l \to 0$ and set $u_l = (\mathbb{1}_{K_l} u)_{\varepsilon_l}$. Then for $K \Subset U$

$$\|u_l - u\|_{L^1(K)} \overset{l \gg 1}{=} \|u_{\varepsilon_l} - u\|_{L^1(K)} \underset{l\to\infty}{\overset{\text{HW Q1.4}}{\to}} 0$$

and

$$\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} u_l - v \right\|_{L^1(K)} \overset{l \gg 1}{=} \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} u_{\varepsilon_l} - v \right\|_{L^1(K)} \overset{\text{Theorem 66}}{=} \|(\partial^\alpha u)_{\varepsilon_l} - v\|_{L^1(K)}$$

$$= \|v_{\varepsilon_l} - v\|_{L^1(K)} \underset{l\to\infty}{\overset{\text{HW Q1.4}}{\to}} 0$$

" $\Longleftarrow$ " For $\varphi \in \mathcal{D}$ compute

$$\int_U u \frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha}\,dx \overset{\varphi \text{ comp supp}}{=} \lim_{l\to\infty} \int_U u_l \frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha}\,dx = (-1)^{|\alpha|} \lim_{l\to\infty} \int_U \frac{\partial^{|\alpha|} u_l}{\partial x^\alpha} \varphi\,dx$$

$$\overset{\varphi \text{ comp supp}}{=} (-1)^{|\alpha|} \int_U v\varphi\,dx,$$

i.e $\partial^\alpha u = v$ weakly.

□

Remark 69

1. The right-hand side of Theorem 68 is an alternate definition of weak derivatives
2. Before the wide use of mollification only $\Longleftarrow$ was known and the right-hand side was called a strong derivatives.
3. There is a famous paper titled "H=W" (Meyers and Serrin 1964) that showed $\Longrightarrow$.
4. Since convergence in the norm implies the existence of an almost everywhere converging subsequence we can assume a.e. convergence on the right-hand side.
5. The same theorem holds in L^p_{loc} for $p < \infty$. If $p = \infty$ the convergence is a.e. and the sequence is locally uniformly bounded but we do not have L^∞ convergence.
6. Instead of $u_l \in C^\infty$ it is enough that $\frac{\partial^{|\beta|} u_l}{\partial x^\beta} \in C^0$ for all $\beta \leq \alpha$.
7. Instead of the L^1_{loc} convergence it is enough if $u_l \to u$ and $\frac{\partial^{|\alpha|} u}{\partial x^\alpha} \to v$ in the (much weaker) sense of distributions to have $\partial^\alpha u = v$ in the sense of distributions.
8. This theorem is crucial in establishing calculus rules for weak derivatives. The idea is to use the rules for smooth functions and take the limit. The assumptions need to be checked though!

11.1 Product rule and chain rule

Theorem 70 (Product Rule)
For $U \subset \mathbb{R}^n$ open, $u, v \in L^1_{\text{loc}}(U)$, $i \in 1, \ldots, n$ if $\partial_i u, \partial_i v \in L^1_{\text{loc}}$ exist and there exists $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that $u, \partial_i u \in L^p_{\text{loc}}$ and $v, \partial_i v \in L^q_{\text{loc}}$, then

$$\partial_i(uv) = v\partial_i u + u\partial_i v$$

Proof
First $1 < p, q < \infty$
By Theorem 68 + remark there exists $u_l, v_l \in C^\infty$ such that $u_l \to u$, $\frac{\partial u_l}{\partial x_i} \to u$ in L^p_{loc}, and $v_l \to v$, $\frac{\partial v_l}{\partial x_i} \to v$ in L^q_{loc}. Therefore for $K \Subset U$

$$\begin{aligned} \|u_l v_l - uv\|_{L^1(K)} &\leq \|u_l v_l - uv_l\|_{L^1(K)} + \|uv_l - uv\|_{L^1(K)} \\ &\overset{\text{Hölder}}{\leq} \underbrace{\|u_l - u\|_{L^p(K)}}_{\to 0} \underbrace{\|v_l\|_{L^q(K)}}_{\to \|v\|_{L^q(K)} < \infty} + \underbrace{\|u\|_{L^p(K)}}_{<\infty} \underbrace{\|v_l - v\|_{L^q(K)}}_{\to 0} \\ &\to 0 \end{aligned} \tag{11.1}$$

and

$$\frac{\partial}{\partial x_i}(u_l v_l) = \frac{\partial u_l}{\partial x_i} v_l + u_l \frac{\partial v_l}{\partial x_i} \overset{\heartsuit}{\to} (\partial_i u)v + u(\partial_i v)$$

in L^1_{loc}. Where $\heartsuit$ works exactly as (11.1) for each summand. Therefore Theorem 68 yields $\partial_i(uv) = v\partial_i u + u\partial_i v$.

For $p = 1, q = \infty$ We don't have $v_l \to v$ in L^∞_{loc} but merely a.e. and for given $K \Subset U$ uniform bound M. Then we have the similar

$$\begin{aligned}\|u_l v_l - uv\|_{L^1(K)} &\leq \|u_l v_l - uv_l\|_{L^1(K)} + \|uv_l - uv\|_{L^1(K)} \\ &\overset{\text{Hölder}}{\leq} \underbrace{\|u_l - u\|_{L^1(K)}}_{\to 0} \underbrace{\|v_l\|_{L^\infty(K)}}_{\leq M < \infty} + \int_K \underbrace{|u|\,|v_l - v|}_{\leq 2M|u| \in L^1(K)}\, dx\end{aligned}$$

Therefore dominated convergence allows us to pass to the limit and implies

$$\int_K |u| \underbrace{|v_l - v|}_{\to 0 \text{ a.e.}} \to 0.$$

Similar for the actual derivatives. $\square$

Theorem 71 (Chain Rule)
For $U \subset \mathbb{R}^n$ open, $u \in L^1(U, \mathbb{R}^N)$, $\nabla u \in L^1(U, \mathbb{R}^{N\times n})$ and globally Lipschitz $g \in C^1(\mathbb{R}^N, \mathbb{R}^M)$ one has $g(u(\cdot)) \in L^1(U, \mathbb{R}^M), \nabla_x g(u(x)) \in L^1(U, \mathbb{R}^{M\times n})$ and

$$\partial_{x_j} g(u(x)) = \sum_{i=1}^N \frac{\partial g(u)}{\partial u^i}(x) \partial_{x_j} u^i(x)$$

for a.e. $x \in U$ and $j = 1, \ldots, n$ or as matrix

$$\underbrace{\nabla_x g(u(x))}_{M\times n} = \underbrace{\nabla_u g(u(x))}_{M\times N} \underbrace{\nabla_x u}_{N\times n}.$$

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Proof
Idea: First show that everything is sufficiently regular, get sequence for u, then show that this sequence converges to the right thing.
For $x \in U$

$$|g(u(x))| \leq |g(u(x)) - g(0)| + |g(0)| \leq \underbrace{(\operatorname{Lip} g)}_{\leq c} \underbrace{|u(x)|}_{\in L^1_{\text{loc}}} + \underbrace{|g(0)|}_{\leq c \in L^\infty_{\text{loc}} \subset L^1_{\text{loc}}}$$

showing $g(u) \in L^1_{\text{loc}}(U, \mathbb{R}^M)$.
Since $\nabla g \in C^1(\mathbb{R}^N, \mathbb{R}^M)$ and $|u(x)| < \infty$ a.e. we have

$$\nabla_u g(u(x)) \in L^\infty(U, \mathbb{R}^{M\times N}).$$

and therefore

$$\|\nabla_u g(u(\cdot))\nabla_x u(\cdot)\|_{L^1_{\text{loc}}(U)} \le \|\nabla_u g(u(\cdot))\|_{L^\infty(U)}\|\nabla_x u(\cdot)\|_{L^1_{\text{loc}}(U)} < \infty$$

implying

$$\nabla_u g(u(x))\nabla_x u \in L^1_{\text{loc}}(U).$$

concluding the regularity estimates.
By Theorem 68 + Remark we can approximate u by u_l such that

$$u_l \to u, \quad \nabla u_l \to \nabla u \text{ in } L^1_{\text{loc}} \tag{11.2}$$

and a.e. in U. Then for $K \Subset U$

$$\|g(u_l(\cdot)) - g(u(\cdot))\|_{L^1(K)} \le (\operatorname{Lip} g)\|u_l - u\|_{L^1(K)} \to 0$$

showing $g(u_l) \to g(u)$ in $L^1_{\text{loc}}(U, \mathbb{R}^M)$. And we can calculate

$$\begin{aligned}
\frac{\partial}{\partial x_j}(g(u_l(x))) &= \sum_{i=1}^N \frac{\partial}{\partial u_l^i} g(u_l(x)) \frac{\partial}{\partial x_j} u_l^i(x) \\
&= \sum_{i=1}^N \underbrace{\underbrace{\frac{\partial}{\partial u_l^i} g}_{\in C^0} \underbrace{(u_l(x))}_{\in C^\infty}}_{\le c} \underbrace{\left(\frac{\partial}{\partial x_j} u_l^i(x) - \partial_{x_j} u^i(x)\right)}_{\to 0 \in L^1_{\text{loc}} \text{ by } (11.2)} \\
&\quad + \sum_{i=1}^N \underbrace{\frac{\partial}{\partial u_l^i} g}_{C^0} \underbrace{(u_l(x))}_{\substack{\text{a.e.}\to u,\\ \text{uniformly bounded}}} \underbrace{\partial_{x_j} u^i(x)}_{\in L^1_{\text{loc}}} \\
\text{dom conv} \implies \quad &\to \sum_{i=1}^N \frac{\partial}{\partial u^i} g(u(x)) \partial_{x_j} u^i(x)
\end{aligned}$$

in $L^1_{\text{loc}}(U)$. Therefore Theorem 68 yields

$$\partial_{x_j}(g(u(x))) = \sum_{i=1}^N \frac{\partial}{\partial u^i} g(u(x)) \partial_{x_j} u^i(x).$$

□

11.2 Definition of Sobolev Spaces

Definition 72
Let $U \subset \mathbb{R}^n$ be open, $m \in \mathbb{N}_0$, $p \in [1, \infty]$

1. The Sobolev $W^{m,p}(U)$-norm is defined by

$$\|u\|_{W^{m,p}(U)} = \begin{cases} \left(\sum_{|\alpha|<m} \int_U |\partial^\alpha u|^p \, dx\right)^{\frac{1}{p}} & \text{if } p < \infty \\ \max_{|\alpha|\le m} \operatorname{ess\,sup}_U |\partial^\alpha u| & \text{if } p = \infty \end{cases}$$

2. The Sobolev space $W^{m,p}$ is defined by

$$W^{m,p}(U, \mathbb{R}^N) = \left\{ u \in L^1_{\text{loc}}(U, \mathbb{R}^N) \,\middle|\, \|u\|_{W^{m,p}(U)} < \infty \right\}$$

3. The subspace $W_0^{m,p}$ of functions with zero boundary values is the closure of $\mathcal{D}(U, \mathbb{R}^n)$ with respect to
   - $\|\cdot\|_{W^{m,p}}$ if $p < \infty$
   - the convergence $\partial^\alpha u_k \to \partial^\alpha u$ uniformly in U for $|\alpha| < m$ $\mathcal{L}^n$ a.e. and uniformly bounded in U for $|\alpha| = m$ if $p = \infty$

4. The localized Sobolev space is

$$W^{m,p}_{\text{loc}}(U, \mathbb{R}^N) = \left\{ u \in L^1_{\text{loc}}(U, \mathbb{R}^N) \,\middle|\, u|_V \in W^{m,p}(V, \mathbb{R}^N) \text{ for all } V \Subset U \right\}$$

Remark 73 ((Functional Analysis) Facts About Sobolev Spaces)

1. $W^{m,p}$ and $W_0^{m,p}$ are complete (normed) spaces, so Banach spaces.
2. $W^{m,p}$ and $W_0^{m,p}$ are separable for $p \in [1, \infty)$
3. $W^{m,p}$ and $W_0^{m,p}$ are reflexive for $p \in (1, \infty)$
4. In a reflexiv Banach space every bounded sequence x_k has a weakly convergent subsequence $x_{k_l} \overset{l\to\infty}{\rightharpoonup} x \in X$
5. For $p = 2$, $W^{m,2}$ is even an inner product space with the inner product

$$(u, v)_{W^{m,2}} = \sum_{|\alpha|\le m} \int \partial^\alpha u \cdot \partial^\alpha v \, dx$$

   so it is even a Hilbert space. One usually writes $H^m = W^{m,2}$, $H^m_{\text{loc}} = W^{m,2}_{\text{loc}}$, $H^m_0 = W^{m,2}_0$
6. The zero boundary conditions hold in a weak sense.
7. If $U = \mathbb{R}^n$, then $W^{m,p}$ functions decay for $p < \infty$, i.e.

$$W^{m,p}(\mathbb{R}^n, \mathbb{R}^N) = W_0^{m,p}(\mathbb{R}^n, \mathbb{R}^N) \quad \text{for } 1 \le p < \infty$$

Definition 74 (Negative Sobolev space)
For $U \subset \mathbb{R}^n$ open, $m \in \mathbb{N}$, $q \in [1, \infty]$

$$W^{-m,q}(U, \mathbb{R}^N) := \left\{ \sum_{|\alpha| \leq m} \partial^\alpha v_\alpha \in \mathcal{D}^\star \;\middle|\; (v_\alpha)_{|\alpha| \leq m} \in L^q(U, \mathbb{R}^N)^{L_m} \right\} \subset \mathcal{D}^\star(U, \mathbb{R}^n)$$

where $L_m = \{\#\alpha \,|\, |\alpha| \leq m\}$ and its norm is given by

$$\|T\|_{W^{-m,q}} := \inf \left\{ \left(\sum_{|\alpha| \leq m} \int_U |v_\alpha|^q \right)^{\frac{1}{q}} \;\middle|\; T = \sum_{|\alpha| \leq m} \partial^\alpha v_\alpha, v_\alpha \in L^q(U, \mathbb{R}^N) \right\}.$$

the inf is necessary since $T = \sum_\alpha \partial^\alpha v_\alpha$ is not unique. One can just add functions in the other direction.

Remarks 75
For $U \subset \mathbb{R}^n$ open, $m \in \mathbb{N}_0$, $p \in [1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$

1.
$$\left(W_0^{m,p}(U, \mathbb{R}^N)\right)^\star \cong W^{-m,q}(U, \mathbb{R}^N)$$

   and

$$\begin{aligned} & u_k \rightharpoonup u && \text{weakly in } W^{m,p}(U, \mathbb{R}^N) \\ \iff & \partial^\alpha u_k \rightharpoonup \partial^\alpha u && \text{weakly in } L^p(U, \mathbb{R}^N)\ \forall |\alpha| \leq m \\ \iff & \int_U \partial^\alpha u_k v \, dx \to \int_U \partial^\alpha u v \, dx && \forall v \in L^q(U, \mathbb{R}^N), |\alpha| \leq m \end{aligned}$$

2. (Meyer-Serrin theorem)
   (a) $C^\infty(U, \mathbb{R}^N) \cap W^{m,p}(U, \mathbb{R}^N)$ is dense in $W^{m,p}(U, \mathbb{R}^N)$
   (b) $C_c^\infty(U, \mathbb{R}^N)$ is dense in $W^{-m,p}(U, \mathbb{R}^N)$

   This allows global approximation of $W^{m,p}$ functions by smooth functions. This is part of the "H=W" paper Meyers and Serrin 1964

Lecture 15 (March 04)

12 Sobolev Embedding

Theorem 76 (Sobolev Embedding/Inequality)
Let $U \subset \mathbb{R}^n$ be open.

(i) Gagliardo-Nirenberg Type
For $p \in [1, n)$ and $p^\star = \frac{np}{n-p} > p$

(a) one has

$$W_0^{1,p}(U,\mathbb{R}^N) \subset L^{p^\star}(U,\mathbb{R}^N)$$

and

$$\|u\|_{L^{p^\star}(U)} \leq C(n,p)\|\nabla u\|_{L^p(U)}$$

for every $u \in W_0^{1,p}(U,\mathbb{R}^N)$

(b) If U has a bounded and Lipschitz boundary one has

$$W^{1,p}(U,\mathbb{R}^N) \subset L^{p^\star}(U,\mathbb{R}^N)$$

and

$$\|u\|_{L^{p^\star}(U)} \leq C(n,p,U)\|u\|_{W^{1,p}(U)}$$

for every $u \in W^{1,p}(U,\mathbb{R}^N)$

(ii) Morrey Type
For $p \in (n,\infty]$

(a) one has

$$W_0^{1,p}(U,\mathbb{R}^N) \subset C^{0,1-\frac{n}{p}}(U,\mathbb{R}^N)$$

and

$$[u]_{C^{0,1-\frac{n}{p}}(U)} \leq C(n,p)\|\nabla u\|_{L^p(U)}$$

for every $u \in W_0^{1,p}(U,\mathbb{R}^N)$

(b) If U has a bounded and Lipschitz boundary one has

$$W^{1,p}(U,\mathbb{R}^N) \subset C^{0,1-\frac{n}{p}}(U,\mathbb{R}^N)$$

and

$$\|u\|_{C^{0,1-\frac{n}{p}}(U)} \leq C(n,p,U)\|u\|_{W^{1,p}(U)}$$

for every $u \in W^{1,p}(U,\mathbb{R}^N)$

Remarks 77

1. In short

$$\begin{array}{rcc} & p<n & p>n \\ u|_{\partial U}=0 & \|u\|_{\frac{np}{n-p}} \lesssim \|\nabla u\|_p & [u]_{C^{0,1-\frac{n}{p}}} \lesssim \|\nabla u\|_p \\ \partial U \text{ bounded} & \|u\|_{\frac{np}{n-p}} \lesssim \|u\|_{W^{1,p}} & \|u\|_{C^{0,1-\frac{n}{p}}} \lesssim \|u\|_{W^{1,p}} \end{array}$$

and

$$p^\star = \frac{np}{n-p} > p \quad \Leftrightarrow \quad p = \frac{np^\star}{n+p^\star} < n$$

2. U has a Lipschitz boundary if for every $x \in \partial U$ there exist neighbourhoods U of x and V of 0 in $\mathbb{R}^n$ and a Bi-Lipschitz map $\Phi : U \to V$ such that

$$\begin{aligned} x \in U \quad &\Leftrightarrow \quad \Phi_1(x) > 0 \\ x \in \partial U \quad &\Leftrightarrow \quad \Phi_1(x) = 0 \\ x \notin \overline{U} \quad &\Leftrightarrow \quad \Phi_1(x) < 0 \end{aligned}$$

So it can be locally straightened. Draw picture!

3. The Hölder semi-norm and norm are defined by

$$[u]_{C^{0,\alpha}(U)} = \sup_{\substack{x,y\in U \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$
$$\|u\|_{C^{0,\alpha}(U)} = \sup_{x\in U} |u(x)| + [u]_{C^{0,\alpha}(U)}$$

4. The decisive point of part (i) is that $p^\star > p$. So if u is integrable and has a weak derivative in this space it is automatically better integrable.

5. Part (ii) is meant in the way that every $u \in W^{1,p}_{(0)}$ has a Hölder continuous representative.

6. Both together applied repeatedly say that if we have enough weak regularity and integrability, then we have classical regularity.

7. $p^\star$ and $1 - \frac{n}{p}$ are optimal. This can be checked by $u(x) = |x - x_0|^s$ for $s < 0$.

8. m times repeated application yields for $l, m \in \mathbb{N}_0$, $p^\star(m) = p^{\overbrace{\star\cdots\star}^{m \text{ times}}} = \frac{np}{n-mp}$

$$\begin{array}{rcc} & mp<n & (m-1)p<n<mp \\ u|_{\partial U}=0 & \|u\|_{W^{l,p^\star(m)}} \lesssim \|\nabla^m u\|_{W^{l,p}} & [u]_{C^{l,m-\frac{n}{p}}} \lesssim \|\nabla^m u\|_{W^{l,p}} \\ \partial U \text{ bounded} & \|u\|_{W^{l,p^\star(m)}} \lesssim \|u\|_{W^{l+m,p}} & \|u\|_{C^{l,m-\frac{n}{p}}} \lesssim \|u\|_{W^{l+m,p}} \end{array}$$

9. For $mp = n$ there is no embedding of type $W^{m,p}_{(0)} \subset L^\infty$ (counterexample $m = 1, p = n \geq 2, u(x) = \log(\log(\frac{e}{|x|}))$ on $B_1(0)$).

   Instead one only has

   $$\|u\|_{L^s} \lesssim \|u\|_{W^{m,p}} \qquad \text{for all } s < \infty \text{ on bounded } U$$

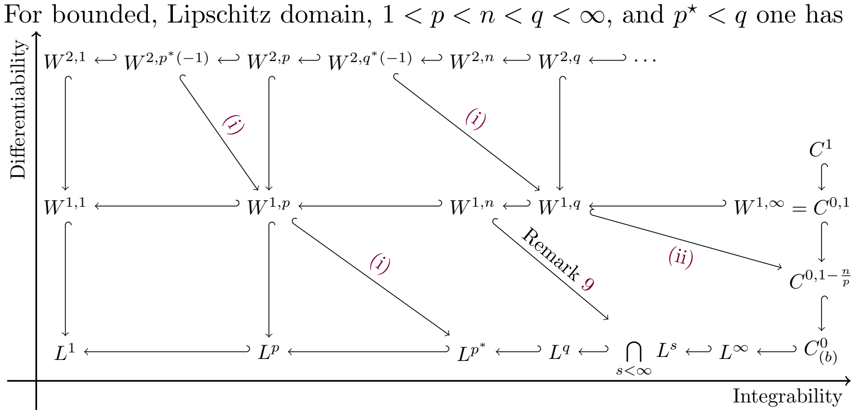
   Except for $p = 1$, $m = n$ one indeed has

   $$W^{n,1} \subset C^0$$

   and if either $u|_{\partial U} = 0$ or U bounded

   $$\|u\|_\infty \lesssim \|u\|_{W^{n,1}}.$$

10. For bounded, Lipschitz domain, $1 < p < n < q < \infty$, and $p^\star < q$ one has



   The $\hookleftarrow$ by Hölder and the $\hookdownarrow$ are trivial. For $W^{1,\infty} = C^{0,1}$ see for example Juha Heinonen - Lectures on Lipschitz Analysis, Theorem 4.1.

11. Dualizing (i)a gives embeddings for negative orders

   $$L^q \subset W^{-m,q^\star(m)}$$

   for $m \in \mathbb{N}$, $1 < q \leq \frac{n}{m}$

   $$\mathrm{RM} \subset W^{-m,s}$$

   for $n \leq m \in \mathbb{N}$, $1 \leq s \leq \infty$ or $n > m \in \mathbb{N}$, $1 \leq s < \frac{n}{n-m}$ and generally

   $$W^{l+m,p} \subset W^{l,p^\star(m)}$$

   for $l \in \mathbb{Z}, m \in \mathbb{N}, 1 < p < \frac{n}{m}$ if ∂U lipschitz and bounded

Proof

(i)a: For $u \in C_c^1(\mathbb{R}^n, \mathbb{R}^N)$

$$u(x) = \int_{-\infty}^{x_i} \partial_i u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)\, dy_i$$

for $i = 1, \ldots, n$ implying

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |\nabla u(x_1, \ldots, y_i, \ldots, x_n)|\, dy_i \right)^{\frac{1}{n-1}}$$

where every factor is independent of x_i. Integrating and using the more general Hölder inequality

$$\left\| \prod_i^k f_i \right\|_{L^1} \leq \prod_i^k \|f_i\|_{L^{p_i}}, \text{ where } \sum_i \frac{1}{p_i} = 1$$

yields

$$\begin{aligned}
&\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}}\, dx_1 \\
&\quad \leq \int_{\mathbb{R}} \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |\nabla u(x_1, \ldots, y_i, \ldots, x_n)|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&\quad = \left(\int_{\mathbb{R}} |\nabla u|\, dy_1 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} |\nabla u|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\
\text{Hölder} \rightarrow &\quad \leq \left(\int_{\mathbb{R}} |\nabla u|\, dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\nabla u|\, dy_i \right)^{\frac{n-1}{n-1}} dx_1 \right)^{\frac{1}{n-1}} \\
&\quad = \left(\int_{\mathbb{R}} |\nabla u|\, dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla u|\, dy_i\, dx_1 \right)^{\frac{1}{n-1}} \qquad (12.1)
\end{aligned}$$

Similar (12.1) yields

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}|u(x)|^{\frac{n}{n-1}}\,dx_1dx_2\\
&\qquad\le \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\nabla u|\,dy_1\right)^{\frac{1}{n-1}}\prod_{i=2}^{n}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\nabla u|\,dy_i\,dx_1\right)^{\frac{1}{n-1}}dx_2\\
&\qquad\overset{i=2\text{ out}}{\le}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\nabla u|\,dy_2\,dx_1\right)^{\frac{1}{n-1}}\\
&\qquad\quad\cdot\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\nabla u|\,dy_1\right)^{\frac{1}{n-1}}\prod_{i=3}^{n}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\nabla u|\,dy_i\,dx_1\right)^{\frac{1}{n-1}}dx_2\\
&\qquad\overset{\text{Hölder}}{\le}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\nabla u|\,dx_1\,dy_2\right)^{\frac{1}{n-1}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|\nabla u|\,dy_1dx_2\right)^{\frac{1}{n-1}}\\
&\qquad\quad\cdot\prod_{i=3}^{n}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}|\nabla u|\,dy_i\,dx_1dx_2\right)^{\frac{1}{n-1}}
\end{aligned}$$

Inductively this yields

$$\begin{aligned}
\|u\|_{L^{\frac{n}{n-1}}}^{\frac{n}{n-1}} &= \int_{\mathbb{R}^n}|u(x)|^{\frac{n}{n-1}}dx\\
&\le \prod_{i=1}^{n}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}|\nabla u|dx_1\dots dx_{i-1}dx_{i+1}\dots dx_n\,dy_i\right)^{\frac{1}{n-1}}\\
&= \left(\int_{\mathbb{R}^n}|\nabla u|dx\right)^{\frac{n}{n-1}} = \|\nabla u\|_{L^1}^{\frac{n}{n-1}} \qquad (12.2)
\end{aligned}$$

proving the case $p=1$ since $p^\star = \frac{np}{n-p} = \frac{n}{n-1}$ for $u \in C_c^\infty$.

For $1<p<n$ applying (12.2) to $v=|u|^\gamma$ with $\gamma = \frac{p(n-1)}{n-p} > 1$ one has

$$\frac{\gamma n}{n-1} = \frac{p(n-1)}{n-p}\frac{n}{n-1} = \frac{pn}{n-p} = p^\star \qquad (12.3)$$

$$(\gamma-1)\frac{p}{p-1} = \frac{pn-p-(n-p)}{n-p}\frac{p}{p-1} = \frac{n(p-1)}{n-p}\frac{p}{p-1} = p^\star \qquad (12.4)$$

$$\gamma - p^\star\frac{p-1}{p} = \frac{p(n-1)}{n-p} - \frac{np}{n-p}\frac{p-1}{p} = \frac{pn-p-np+n}{n-p} = 1 \qquad (12.5)$$

and

$$\begin{aligned}
\|u\|^{\gamma}_{L^{p^\star}} &\overset{(12.3)}{=} \left(\int_{\mathbb{R}^n} |u|^{\gamma\frac{n}{n-1}}\right)^{\frac{n-1}{n}} = \|v\|_{L^{\frac{n}{n-1}}} \\
&\overset{(12.2)}{\leq} \|\nabla v\|_{L^1} = \int_{\mathbb{R}^n} |\nabla |u|^{\gamma}|\, dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |\nabla u|\, dx \\
\underset{\frac{1}{p}+\frac{1}{\frac{p}{p-1}}=1}{\text{Hölder}} \rightarrow \quad &\leq \gamma \left(\int_{\mathbb{R}^n} \left(|u|^{\gamma-1}\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{1}{p}} \\
&\overset{(12.4)}{=} \gamma \left(\int_{\mathbb{R}^n} |u|^{p^\star} dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{1}{p}} \\
&= \gamma \|u\|^{p^\star \frac{p-1}{p}}_{L^{p^\star}} \|\nabla u\|_{L^p} \qquad (12.6)
\end{aligned}$$

Therefore

$$\|u\|_{L^{p^\star}} \overset{(12.5)}{=} \|u\|^{\gamma - p^\star \frac{p-1}{p}}_{L^{p^\star}} \overset{(12.6)}{\leq} \gamma \|\nabla u\|_{L^p} \qquad (12.7)$$

for $u \in C^\infty_c$.

Finally since C^∞_c is dense in $W^{1,p}$ take $u_k \in C^\infty_c$ converging to $u \in W^{1,p}$. Then

$$\|u_m - u_n\|_{L^{p^\star}} \overset{(12.7)}{\leq} \gamma \|\nabla u_m - \nabla u_n\|_{L^p} \to 0$$

implying u_k is Cauchy in $L^{p^\star}$ which is complete, therefore $u_k \to \tilde{u} \in L^{p^\star}$. Wlog $\tilde{u} = u$ (since convergence in $L^{p^\star}$ yields a.e. convergence of a subsequence so take the subsequence if $u \neq \tilde{u}$). Then by (12.7)

$$\|u_k\|_{L^{p^\star}} \leq \gamma \|\nabla u_k\|_{L^p}$$

for all $k \in \mathbb{N}$. Therefore

$$\|u\|_{L^{p^\star}} = \lim_{k\to\infty} \|u_k\|_{L^{p^\star}} \leq \lim_{k\to\infty} \gamma \|\nabla u_k\|_{L^p} \leq \gamma \|\nabla u\|_{L^p}.$$

(i)b: (Sketch) For $U, \tilde{U} \subset \mathbb{R}^n$ open with Lipschitz boundary and $\overline{U} \subset U$ there exists an continuous extension (Stein extension) $E : W^{m,p}(U, \mathbb{R}^N) \to W^{m,p}(\mathbb{R}^n, \mathbb{R}^N)$ such that

$$Eu|_U = u, \qquad Eu|_{\mathbb{R}^n \setminus \tilde{U}} = 0.$$

So extending u to Eu yields $Eu \in W^{1,p}_0(\mathbb{R}^n, \mathbb{R}^N)$ and (i)a yields the claim.

(ii)a: We first show for convex and bounded U and $u \in C^\infty$

$$\left| u(x) - \fint_U u\, dy \right| \leq C_{n,p} \frac{(\text{diam} U)^n}{|U|^{1-\frac{p-n}{np}}} \|\nabla u\|_{L^p}. \qquad (12.8)$$

One has

$$
\begin{aligned}
\left|u(x) - \fint_U u\,dy\right| &\leq \frac{1}{|U|}\int |u(x)-u(y)|\,dy \\
y = x + tz \to \qquad &= \frac{1}{|U|}\int_{\partial B_1(0)}\int_0^{r(z)} |u(x)-u(x+\rho z)|\,\rho^{n-1} d\rho dS(z) \\
&\leq \frac{1}{|U|}\int_{\partial B_1(0)}\int_0^{r(z)}\int_0^{\rho} |\nabla u(x+tz)|\,dt\,\rho^{n-1} d\rho dS(z) \\
&\leq \int_0^{\mathrm{diam}U} \rho^{n-1} d\rho \frac{1}{|U|}\int_{\partial B_1(0)}\int_0^{r(z)} |\nabla u(x+tz)|\,dt\,dS(z) \\
&= \frac{(\mathrm{diam}U)^n}{n|U|}\int_{\partial B_1(0)}\int_0^{r(z)} |\nabla u(x+tz)|\underbrace{\frac{t^{n-1}}{|tz|^{n-1}}}_{=1}\,dt\,dS(z) \\
z = \frac{y-x}{t} \to \qquad &= \frac{(\mathrm{diam}U)^n}{n|U|}\int_U |\nabla u(y)| \frac{1}{|y-x|^{n-1}}\,dy \qquad (12.9) \\
\text{Hölder} \to \qquad &\leq \frac{(\mathrm{diam}U)^n}{n|U|}\underbrace{\left(\int_U \frac{1}{|y-x|^{(n-1)\frac{p}{p-1}}}\,dy\right)^{\frac{p-1}{p}}}_{\star} \|\nabla u\|_{L^p}
\end{aligned}
$$

For fixed $|U|$, $\star$ is maximal for $U = B_R(x)$ and then using $p > n$ one can show $\star \leq c_{n,p} R^{\frac{p-n}{p}} = C_{n,p}|U|^{\frac{p-n}{np}}$, proving (12.8).
Next we exploit (12.8) for balls

$$
\begin{aligned}
|u(x_2) - u(x_1)| &\leq \sum_{i=1}^{2}\left|u(x_i) - \fint_{B_{|x_1-x_2|}(\frac{x_1+x_2}{2})} u\right| \\
&\leq C_{p,n}\frac{|x_1-x_2|^n}{|x_1-x_2|^{n-1+\frac{n}{p}}}\|\nabla u\|_{L^p} \\
&\leq C_{p,n}|x_1-x_2|^{1-\frac{n}{p}}\|\nabla u\|_{L^p}
\end{aligned}
$$

proving

$$
[u]_{C^{0,1-\frac{n}{p}}} = \sup_{\substack{x_1,x_2\in U\\ x_1\neq x_2}} \frac{|u(x_2)-u(x_1)|}{|x_2-x_1|^{1-\frac{n}{p}}} \leq C_{p,n}\|\nabla u\|_{L^p}
$$

for balls and $u \in C^\infty$. Approximating again yields this for every $u \in W^{1,p}$.
Additionally

$$
|u(x)| \overset{(12.8)}{\leq} \fint_{B_1} |u|\,dy + C_{n,p}\|\nabla u\|_{L^p} \overset{\text{Hölder}}{\leq} C_{n,p}\|u\|_{W^{1,p}}.
$$

(ii)b: Again the Stein extension and (ii)a yield the claim.

□

Lecture 16 (March 09)

Theorem 78 (Gagliardo-Nirenberg Interpolation)
Let $m \in \mathbb{N}$, $j \in \mathbb{N}_0$, $j < m$, $1 \leq q, r \leq \infty$, $1 \leq p < \infty$ and

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\alpha}{q}, \qquad \frac{j}{m} \leq \alpha \leq 1$$

(a) Then for every $u \in W^{m,r}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$

$$\|\nabla^j u\|_{L^p} \leq c\|\nabla^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}$$

If $j = 0$ and $q = \infty$, then u needs some additional decay assumption such as $u \in L^s(\mathbb{R}^n)$ for some $s < \infty$.

(b) If $U \subset \mathbb{R}^n$ is open and has a bounded and uniformly Lipschitz continuous boundary then for every $u \in W^{m,r}(U) \cap L^q(U)$

$$\|\nabla^j u\|_{L^p} \leq c\|\nabla^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha} + \|u\|_{L^s}$$

for any $1 \leq s \leq \infty$.

If $r > 1$ and $m - j - \frac{n}{r} \in \mathbb{N}_0$, the additional assumption $\alpha < 1$ is needed.

Remarks 79

1. The case $p = 4$, $r = q = 2$, $j = 0$, $m = 1$ is known as Ladyzhenskaya inequality
2. Proofs sketched independently by Gagliardo and Nirenberg (p. 125) but only fully published in Leoni 2017 (although there it is stated a bit differently)

Theorem 80 (Poincaré Inequality)
Let $U \subset \mathbb{R}^n$ be open, $p \in [1, \infty]$.

(i) Zero Boundary Conditions Type If $u \in W_0^{1,p}(U)$ then

$$\|u\|_{L^p} \leq c\|\nabla u\|_{L^p}$$

with

- $c = \frac{l}{2}$ if U is in a strip of width l
- $c = \text{const}(n)|U|^{\frac{1}{n}}$ if $|U| < \infty$.

(ii) Zero Mean Type If $u \in W^{1,p}(U)$

$$\left\| u - \fint_U u \right\|_{L^p} \leq c(n, p, U)\|\nabla u\|_{L^p}$$

if U is connected and bounded with Lischitz boundary.

Remarks 81

1. The optimal Poincaré constant is sometimes of interest and hard to determine. For (i)

   - one has the scaling

   $$c_p^{\text{opt}}(rU) = r c_p^{\text{opt}}(U)$$

   - for $p = 2$ the smallest eigenvalue of $-\Delta$ with 0 boundary conditions is equal to $(c_2^{\text{opt}})^{-2}$

2. - $\|\nabla u\|_{L^p(U)}$ and $\|u\|_{W^{1,p}(U)}$ are equivalent norms on $W_0^{1,p}$
   - $\left|\fint u\right| + \|\nabla u\|_{L^p(U)}$ and $\|u\|_{W^{1,p}(U)}$ are equivalent norms on $W^{1,p}$

3. Iteratively

$$\|u\|_{L^p} \leq c\|\nabla^m u\|_{L^p} \qquad \text{for } u \in W_0^{m,p}$$

Sketch of Proof

(i) - Assume $p < \infty$ ($p = \infty$ similarly) and $U \subset (-\frac{l}{2}, \frac{l}{2}) \times \mathbb{R}^{n-1}$, then for $\varphi \in C_c^\infty(U)$

$$\begin{aligned}
&\int_{U \cap \{x_1 \leq 0\}} |u|^p dx \\
&\overset{\substack{\text{Fundamental}\\ \text{Theorem of}\\ \text{Calculus}}}{=} \int_{-\frac{l}{2}}^0 \int_{\mathbb{R}^{n-1}} \left| \int_{-\frac{l}{2}}^{x_1} \partial_1 u(\tau, x') \, d\tau \right|^p dx' dx_1 \\
&\overset{\text{Hölder}}{\leq} \int_{-\frac{l}{2}}^0 \int_{\mathbb{R}^{n-1}} \int_{-\frac{l}{2}}^{x_1} |\partial_1 u(\tau, x')|^p d\tau \left(\|1\|_{L^{\frac{p}{p-1}}(x_1 + \frac{l}{2})} \right)^p dx' dx_1 \\
&\leq \underbrace{\int_{-\frac{l}{2}}^0 \left(x_1 + \frac{l}{2} \right)^{p-1} dx_1}_{= \frac{1}{p}\left(x_1 + \frac{l}{2}\right)^p |_{-\frac{l}{2}}^0} \int_{U \cap \{x_1 \leq 0\}} |\nabla u|^p \, dx \\
&\leq \left(\frac{l}{2} \right)^p \|\nabla u\|^p_{L^p(U \cap \{x_1 \leq 0\})}
\end{aligned}$$

Similar $x_1 \geq 0$ and then get the inequality for $u \in W_0^{1,p}$ by approximation similar to Sobolev embedding.

- For $|U| = 1$ by rescaling and $n \geq 2$ given p choose $q = \frac{pn}{n+p}$ such that $q < n$ with $q \leq p \leq q^\star$ and get

$$\|u\|_{L^p} \overset{\text{Hölder}}{\leq} \|u\|_{L^{q^\star}} \overset{\text{Sobolev (i)a}}{\leq} c(n,p)\|\nabla u\|_{L^q} \overset{\text{Hölder}}{\leq} c(n,p)\|\nabla u\|_{L^p}$$

(ii) At first consider a convex domain, then by (12.9) get

$$\int_U \left| u - \fint_U u \right|^p dx \leq \frac{(\text{diam}U)^n}{n|U|} \int_U |\nabla u(y)| \frac{1}{|y-x|^{n-1}} \, dy$$
$$\text{Hölder} \to \quad \leq \cdots \leq C(U,n,p) \int_U |\nabla u|^p dx \tag{12.10}$$

For arbitrary Lipschitz U cover $U = \cup_{i=1}^k X_i$ by (convex) balls and half balls which are Bilipschitz images of the boundary. Use (12.10) for these and then (carefully) sum over them.

□

Chapter D

Nonlinear First Order Equations

Consider $U \subset \mathbb{R}^n$ open, $F : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ and $g : \Gamma \subset \partial U \to \mathbb{R}$ given (sufficiently smooth) and

$$F(\nabla u, u, \cdot) = 0 \qquad \text{in } U \tag{12.1}$$

$$u = g \qquad \text{on } \Gamma \tag{12.2}$$

We will write

$$F = F(p, z, x) = F(p_1, \dots, p_n, z, x_1, \dots, x_n)$$

so $p = \nabla u(x)$, $z = u(x)$ and for example $\nabla_p F(\nabla u, u, x)$.

These equations are often not solvable but we can get some insights into the solutions.

13 Characteristics

Similar to the transport equation we try to transform the PDE to a system of ODEs. To do this we start at some $x_0 \in \Gamma$ and try to compute u along a curve. Let the curve in $\mathbb{R}^n$ be

$$\bar{x}(s) = (\bar{x}_1(s), \dots, \bar{x}_n(s))$$

then

$$\frac{d}{ds} p_i(\bar{x}(s)) = \nabla p_i \cdot \bar{x}' = \sum_{j=1}^{n} \partial_i \partial_j u(\bar{x}(s)) \bar{x}_i'(s).$$

Differentiating (12.1) yields

$$\begin{aligned} 0 &= \frac{d}{dx_i} F(\nabla u, u, x) \\ &= \nabla_p F(\nabla u, u, x) \cdot \partial_i (\nabla u(x)) + \partial_z F(\nabla u, u, x) \partial_i u + \partial_i F(\nabla u, u, x) \\ &= \sum_{j=1}^{n} \partial_{p_j} F \partial_i \partial_j u + \partial_z F \partial_i u + \partial_i F \end{aligned} \tag{13.1}$$

So if we set the direction of the curve to cancel the second order derivatives by

$$\bar{x}'(s) = \nabla_p F(\nabla u(\bar{x}(s)), u(\bar{x}(s)), \bar{x}(s)) \tag{13.2}$$

then

$$\frac{d}{ds} p_i(\bar{x}(s)) = \sum_{j=1}^{n} \partial_i \partial_j u(\bar{x}(s)) \partial_{p_i} F(\nabla u(\bar{x}(s)), u(\bar{x}(s)), \bar{x}(s)) \tag{13.3}$$

and therefore evaluating (13.1) along the curve, i.e. in $x = \bar{x}(s)$ yields

$$\begin{aligned} 0 &\overset{(13.1)}{=} \sum_{j=1}^{n} \partial_{p_j} F \partial_i \partial_j u + \partial_z F \partial_i u + \partial_i F \\ &\overset{(13.3)}{=} \frac{d}{ds} p|_{x=\bar{x}(s)} + \partial_z F \nabla u|_{x=\bar{x}(s)} + \nabla_x F|_{x=\bar{x}(s)} \end{aligned} \tag{13.4}$$

and

$$\begin{aligned} \frac{d}{ds} u(\bar{x}(s)) &= \nabla u(\bar{x}(s)) \cdot \bar{x}'(s) \\ &\overset{(13.2)}{=} \nabla u(\bar{x}(s)) \cdot \nabla_p F(\nabla u(\bar{x}(s)), u(\bar{x}(s)), \bar{x}(s)) \end{aligned} \tag{13.5}$$

Therefore defining

$$\bar{z}(s) = u(\bar{x}(s)), \qquad \bar{p}(s) = p(\bar{x}(s))$$

yields the $2n + 1$ dimensional ODEs

$$\bar{x}' \overset{(13.2)}{=} \nabla_p F(\bar{p}, \bar{z}, \bar{x}) \tag{13.6}$$

$$\bar{z}' \overset{(13.5)}{=} \bar{p} \cdot \nabla_p F(\bar{p}, \bar{z}, \bar{x}) \tag{13.7}$$

$$\bar{p}' \overset{(13.4)}{=} -\partial_z F(\bar{p}, \bar{z}, \bar{x}) \bar{p} - \nabla_x F(\bar{p}, \bar{z}, \bar{x}) \tag{13.8}$$

The solution of this system is called characteristic and the curve $\bar{x}(s)$ is called the projected characteristic.

Therefore we have proved the following theorem

Theorem 82 (PDE implies ODE)
Let $u \in C^2(U)$ be a solution of (12.1), (12.2) in U. Assume $\bar{x}(\cdot)$ solves (13.6), where $\bar{p}(\cdot) = \nabla u(\bar{x}(\cdot))$ and $\bar{z}(\cdot) = u(\bar{x}(\cdot))$, then $\bar{z}$ solves (13.7) and $\bar{p}$ solves (13.8) for those values such that $\bar{x}(\cdot) \in U$.

The crucial step in reducing from a PDE to an ODE is setting $\bar{x}' = \nabla_p F$ such that the second order derivatives cancel.

Lecture 17 (March 11)

Example 83

1. Linear homogeneous F

$$0 = F(\nabla u, u, x) = b(x) \cdot \nabla u(x) + c(x)u(x) \tag{13.9}$$

where $b : U \to \mathbb{R}^n$, $c : U \to \mathbb{R}$ are given. The characteristic equations are given by

$$\begin{aligned} \bar{x}'(s) &= b(\bar{x}(s)) \\ \bar{z}'(s) &= \bar{p} \cdot b(\bar{x}(s)) \overset{(13.9)}{=} -c(\bar{x}(s))\bar{z}(s) \end{aligned}$$

which is closed and uncoupled from $\bar{p}(s)$.

(a) Explicit F

$$\begin{aligned} x_1 \partial_2 u - x_2 \partial_1 u &= u \qquad \text{in } U = \{x_1, x_2 > 0\} \\ u &= g \qquad \text{on } \Gamma = \{x_1 > 0, x_2 = 0\} \end{aligned}$$

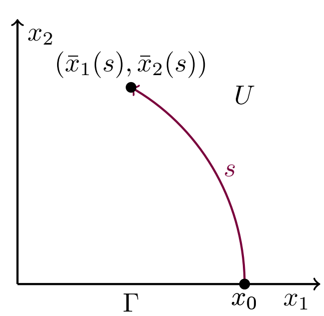
Then $c = -1$, $b = (-x_2, x_1)$ and therefore

$$\begin{aligned} \bar{x}_1'(s) &= -\bar{x}_2(s) \\ \bar{x}_2'(s) &= \bar{x}_1(s) \\ \bar{z}'(s) &= \bar{z}(s) \end{aligned}$$

which is solved by

$$\begin{aligned} \bar{x}_1(s) &= x_0 \cos(s) \\ \bar{x}_2(s) &= x_0 \sin(s) \\ \bar{z}(s) &= z_0 e^s \end{aligned}$$

where $x_0 \geq 0$ and $s \in [0, \frac{\pi}{2}]$ such that $x_1, x_2 > 0$



and therefore

$$x_0^2 = \bar{x}_1^2(s) + \bar{x}_2^2(s), \qquad s = \arctan\left(\frac{x_2}{x_1}\right)$$

Finally

$$\begin{aligned} u(x_1, x_2) = z(s) = z_0 e^s = u(x_0) e^s = g(x_0) e^{\arctan\left(\frac{x_2}{x_1}\right)} \\ = g\left(\sqrt{x_1^2 + x_2^2}\right) e^{\arctan\left(\frac{x_2}{x_1}\right)} \end{aligned}$$

2. Quasilinear F
   For

$$0 = F(\nabla u, u, x) = b(x, u(s)) \cdot \nabla u(x) + c(x, u(x))$$

   exactly as to before

$$\bar{x}'(s) = b(\bar{x}(s), \bar{z}(s)) \tag{13.10}$$
$$\bar{z}'(s) = -c(\bar{x}(s), \bar{z}(s)) \tag{13.11}$$

   but these are coupled and nonlinear.

Note that the boundary condition needs to be provided where the characteristic enters the domain. If $x(s)$ goes along Γ we don't know anything about u in the domain.

14 Boundary Conditions

What are the right boundary conditions to make the (nonlinear first order) PDE solvable?

Straightening the Boundary

Assume ∂U is C^k, then for every $x_0 \in \partial U$ there exists a ball $B_r(x_0)$ such that

$$U \cap B_r(x_0) = \left\{x \in B_r(x_0) \,\middle|\, x_n > \gamma(x_1, \ldots, x_n), \gamma \in C^k\right\}.$$

Define φ by

$$\begin{aligned} y_i &= \varphi_i(x) = x_i && \text{for } i = 1, \ldots, n-1 \\ y_n &= \varphi_n(x) = x_n - \gamma(x_1, \ldots, x_{n-1}) \end{aligned}$$

draw picture
and the inverse ψ by

$$\begin{aligned} x_i &= \psi_i(y) = y_i && \text{for } i = 1, \ldots, n-1 \\ x_n &= \psi_n(y) = y_n + \gamma(y_1, \ldots, y_{n-1}) \end{aligned}$$

Let u be a solution to

$$F(\nabla u, u, \cdot) = 0 \qquad \text{in } U \tag{14.1}$$
$$u = g \qquad \text{on } \Gamma \subset \partial U \tag{14.2}$$

then

$$\begin{aligned} & v(y) = u(\psi(y)) \qquad \text{in } V = \varphi(U) \\ \implies & u(x) = v(\varphi(x)) \qquad \text{in } U = \psi(V) \end{aligned}$$

solves

$$\begin{aligned} 0 &= F(\nabla_x u, u, x) \\ &= F(\nabla_\varphi v(\varphi(x)) \cdot \nabla_x \varphi(x), v(\varphi(x)), x) \\ &= F(\nabla_y v(y) \cdot \nabla \varphi(\psi(y)), v(y), \psi(y)) \\ &=: G(\nabla_y v(y), v(y), y) \qquad \text{in } V \end{aligned}$$

and for the boundary

$$v(y) = h(y) := g(\psi(y))$$

for all $y \in \Delta = \varphi(\Gamma)$, i.e. v solves

$$\begin{aligned} G(\nabla v, v, \cdot) &= 0 \qquad \text{in } V \\ v &= h \qquad \text{on } \Delta \subset \mathbb{R}^{n-1} \times \{y_n = 0\} \end{aligned}$$

which has the same structure as (14.1), (14.2)! This works only locally though.

So from now we can assume that Γ is flat near x_0, i.e. $x_n = 0$.

Compatibility Conditions

We need appropriate initial conditions for the ODEs to be solvable, i.e.

$$\bar{p}(0) = p_0, \qquad \bar{z}(0) = z_0, \qquad \bar{x}(0) = x_0$$

We know $x_0 \in \Gamma$ and $z_0 = u(x_0) = g(x_0)$ so we only need a condition for p_0.

$$u(x_1, \dots, x_{n-1}, 0) = g(x_1, \dots, x_{n-1})$$

and therefore

$$p_{0i}(x_0) = \partial_i u(x_0) = \partial_i g(x_0) \qquad i = 1, \dots, n-1 \tag{14.3}$$

and additionally the PDE should hold, so

$$F(p_0, z_0, x_0) = 0 \tag{14.4}$$

and therefore (14.3) and (14.4) determine the n components of p_0.

Definition 84
The colored equations are called compatibility conditions between the PDE and boundary data. A vector $(p_0, z_0, x_0) \in R^{2n+1}$ that satisfies these is called admissible.

Similarly in a neighbourhood around x_0 the compatibility conditions should be satisfied. Therefore

$$q_i(y) = \partial_{x_i} g(y), \qquad i = 1, \dots, n-1 \tag{14.5}$$

$$F(q(y), g(y), y) = 0 \tag{14.6}$$

should be satisfied for all $y \in \Gamma$ sufficiently close to x_0.

Lemma 85
There exists a unique solution $q(\cdot)$ of (14.5), (14.6) for all $y \in \Gamma$ in a neighbourhood of x_0 provided that

$$\partial_{p_n} F(p_0, z_0, x_0) \neq 0. \tag{14.7}$$

Proof
Let $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$G_i(p, x) = p_i - \partial_i g(x_1, \dots, x_{n-1}) \qquad i = 1, \dots, n-1$$
$$G_n(p, x) = F(p, g(x_1, \dots, x_{n-1}), x)$$

Then

$$\nabla_p G(p_0, x_0) = \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \partial_{p_1} F(p_0, z_0, x_0) & \dots & & \partial_{p_n} F(p_0, z_0, x_0) \end{pmatrix}$$

and therefore $\det \nabla_p G(p_0, x_0) = \partial_{p_n} F(p_0, z_0, x_0) \neq 0$. Therefore the implicit function theorem says that there exists $p(x)$ such that $G(p(x), x) = 0$ provided x is close to x_0. Therefore $G(p(x), x)|_{x \in \Gamma} = 0$ yields the claim. □

(p_0, z_0, x_0) is called non-characteristic if (14.7) holds.

15 Existence of local solutions

Lemma 86 (Local Invertibility)
Assume $\partial_{p_n} F(p_0, z_0, x_0) \neq 0$. The there exists a open interval $I \subset \mathbb{R}$ containing 0, a neighbourhood $W \subset \Gamma \subset \mathbb{R}^{n-1}$ of x_0, a neighbourhood $V \subset \mathbb{R}^n$ of x_0 such that for each $x \in V$ there exists a unique $s \in I$, $y \in W$ such that

$$x = \bar{x}(y, s)$$

where $\bar{x}(y, s)$ is the projected characteristic starting at y, i.e. with $\bar{x}(y, 0) = (y, 0)$. The mapping $x \to s, y$ is C^2.

Proof
We want to again exploit the implicit function theorem. Since $\bar{x}(x_0, 0) = x_0$, the claim follows if $\det \nabla \bar{x}(x_0, 0) \neq 0$. We have

$$\bar{x}(y, 0) = (y, 0)$$

and therefore

$$\partial_{y_i} \bar{x}_j(x_0, 0) = \begin{cases} \delta_{ij} & j = 1, \dots, n-1 \\ 0, & j = n \end{cases}$$

and the characteristic ODE for $x(x_0, s)$ yields

$$\partial_s \bar{x}(x_0, s)|_{s=0} \overset{(13.6)}{=} \nabla_p F(\bar{p}, \bar{z}, \bar{x})|_{s=0} = \nabla_p F(p_0, z_0, x_0)$$

and therefore

$$\nabla \bar{x}(x_0, 0) = \begin{pmatrix} 1 & & 0 & \partial_{p_1} F(p_0, z_0, x_0) \\ & \ddots & & \vdots \\ 0 & & 1 & \vdots \\ 0 & \dots & 0 & \partial_{p_n} F(p_0, z_0, x_0) \end{pmatrix}$$

implying $\det \bar{x}(x_0, 0) \neq 0$ by the non-characteristic assumption. □

Since for each $x \in V$ there is a unique s, y such that

$$x = \bar{x}(y, s) \tag{15.1}$$

we can define

$$y = \bar{y}(x), \qquad s = \bar{s}(x)$$

and

$$u(x) = \bar{z}(\bar{y}(x), \bar{s}(x)) \tag{15.2}$$

Theorem 87 (Local Existence)
Assume (p_0, z_0, x_0) is admissible (Definition 84) and $\partial_{p_n} F(p_0, z_0, x_0) \neq 0$. The function defined in (15.2) is C^2 and solves

$$\begin{aligned} F(\nabla u, u, \cdot) &= 0 && \text{in } V \\ u &= g && \text{on } \Gamma \cap V \end{aligned}$$

Proof

1. Fix y close to x_0 and let $\bar{x}(y, s), \bar{z}(y, s), \bar{p}(y, s)$ solve the characteristic ODEs with starting point $\bar{x}(y, 0) = y$.

2. Then by construction we claim

$$f(y,s) = F(\bar{p}(y,s), \bar{z}(y,s), \bar{x}(y,s)) = 0 \tag{15.3}$$

This holds since by (14.6)

$$f(y,0) = F(\bar{p}(y,0), \bar{z}(y,0), \bar{x}(y,0)) = F(q(y), g(y), y) \overset{(14.6)}{=} 0$$

and by the characteristic equations

$$\begin{aligned}\frac{d}{ds}f(y,s) &= \nabla_p F \cdot \frac{d}{ds}\bar{p} + \partial_z F \frac{d}{ds}\bar{z} + \nabla_x F \cdot \frac{d}{ds}\bar{x} \\ &\overset{(13.6)-(13.8)}{=} -\nabla_p F \cdot \partial_z F \bar{p} - \nabla_p F \cdot \nabla_x F + \partial_z F \bar{p} \cdot \nabla_p F + \nabla_x F \cdot \nabla_p F \\ &= 0\end{aligned}$$

3. By (15.1)-(15.2) and Lemma 86

$$0 \overset{(15.3)}{=} F(\bar{p}(y,s), \bar{z}(y,s), \bar{x}(y,s)) \overset{(15.1)-(15.2)}{=} F(\bar{p}(y,s), u(x), x)$$

4. It is left to show that $\bar{p}(x) = \bar{p}(y,s) = \nabla u(x)$. To do this first establish

$$\bar{z}' \overset{(13.7)}{=} \bar{p} \cdot \nabla_p F \overset{(13.6)}{=} \bar{p} \cdot \bar{x}' \tag{15.4}$$

and

$$\nabla_y \bar{z}(y,s) = \sum_j \bar{p}_j(y,s) \nabla_y \bar{x}_j(y,s) \tag{15.5}$$

which can be shown similar to $f(y,s) = 0$.

To explicitly prove (15.5) define

$$r(s) = \nabla_y \bar{z}(y,s) - \sum_j \bar{p}_j(y,s) \nabla_y \bar{x}_j(y,s). \tag{15.6}$$

Then by the compatibility conditions in the neighbourhood

$$\begin{aligned}r(0) &= \nabla_y \bar{z}(y,0) - \sum_j \bar{p}_j(y,0) \nabla_y \bar{x}_j(y,0) \\ &= \nabla_y g(y) - \sum_j q_j(y) \underbrace{\nabla_y y_j}_{\delta_{ij}} \\ &= \nabla_y g(y) - q_j(y) \overset{(14.5)}{=} 0\end{aligned} \tag{15.7}$$

Computing the gradient of (15.4)

$$\nabla \bar{z}' = \sum_j \nabla \bar{p}_j \bar{x}_j' + \sum_j \bar{p}_j \nabla \bar{x}_j' \tag{15.8}$$

which yields

$$\begin{aligned} r_i' &\overset{(15.6)}{=} \nabla_y \bar{z}' - \sum_j \bar{p}_j' \nabla \bar{x}_j - \sum_j \bar{p}_j \nabla \bar{x}_j' \\ &\overset{(15.8)}{=} - \sum_j \bar{p}_j' \nabla \bar{x}_j + \sum_j \nabla \bar{p}_j \bar{x}_j' \\ &\overset{(13.8),(13.6)}{=} \sum_j \partial_{x_j} F \nabla \bar{x}_j + \sum_j \partial_z F \bar{p}_j \nabla \bar{x}_j + \sum_j \nabla \bar{p}_j \partial_{p_j} F \end{aligned} \tag{15.9}$$

Taking the gradient of (15.3)

$$0 = \sum_j \partial_{p_j} F \nabla_y \bar{p}_j + \partial_z F \nabla_y \bar{z} + \sum_j \partial_{x_j} F \nabla \bar{x}_j$$

such that by (15.9)

$$\begin{aligned} r' &= \sum_j \partial_z F \bar{p}_j \nabla \bar{x}_j - \partial_z F \nabla_y \bar{z} = \partial_z F \left(\sum_j \bar{p}_j \nabla_y \nabla \bar{x}_j - \nabla_y \bar{z} \right) \\ &= -\partial_z F r \end{aligned}$$

implying

$$r(s) = r(0) e^{-\int_0^s \partial_z F \ d\tau} \overset{(15.7)}{=} 0$$

5. Finally

$$\begin{aligned} \nabla_x u(x) &\overset{(15.2)}{=} \nabla_x \bar{z}(y, s) = \nabla_y \bar{z} \cdot \nabla_x y + \bar{z}' \nabla_x s \\ &\overset{(15.4),(15.5)}{=} \sum_j \bar{p}_j \nabla_y \bar{x}_j \cdot \nabla_x y + \bar{p} \cdot \bar{x}' \nabla_x s \\ &= \sum_j \bar{p}_j \left(\nabla_y \bar{x}_j (y(x), s(x)) \cdot \nabla_x y + \bar{x}_j' \nabla_x s(y(x), s(x)) \right) \\ &= \sum_j \bar{p}_j \nabla_x x_j = \bar{p} \end{aligned}$$

□

Remark 88 (Non-Flat Boundaries)
For non flat boundaries the non-characteristic condition $\partial_{p_n} F(p_0, z_0, x_0) \neq 0$ is

$$\nu(x_0) \cdot \nabla_p F(p_0, z_0, x_0) \neq 0,$$

where $\nu(x_0)$ is the outward normal at x_0.
The compatibility conditions (14.3) then have to hold in the tangent plane.

16 Conservation Laws

We will study the PDE

$$\partial_t u + \partial_x F(u) = 0 \qquad \text{in } \mathbb{R} \times (0, \infty) \tag{16.1}$$

$$u = g \qquad \text{on } \mathbb{R} \times \{t = 0\} \tag{16.2}$$

where $F : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given.

These are really common since they conserve the quantity u (usually a concentration or density) on a domain when taking into consideration the out-/inflow flux.

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b u_t(x,t)\,dx = -\int_a^b \partial_x F(u)\,dx = F(u(a)) - F(u(b)),$$

where $F(u(a))$ is the inflow on the left-hand side and $F(u(b))$ is the outflow on the right-hand side.

Similar in n dimensions and $F : \mathbb{R} \to \mathbb{R}^n$,

$$\partial_t u + \nabla \cdot F(u) = 0$$

and $\Omega \subset U$ arbitrary

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = \int_\Omega u_t(x,t)\,dx = -\int_\Omega \nabla \cdot F(u)\,dx = -\int_{\partial\Omega} \nu \cdot F(u(x))\,dS(x)$$

so the change in the domain is equal to the negative flow over the boundary.

Note that can rewrite the PDE as

$$\partial_t u + F'(u)u_x = 0.$$

16.1 Weak Solutions

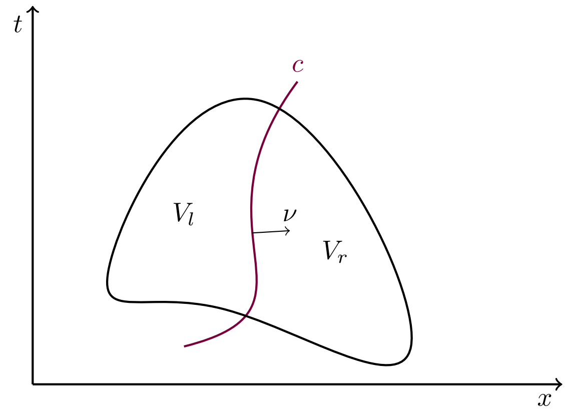
Let $\varphi \in C^\infty(\mathbb{R} \times [0, \infty))$ have compact support (potentially $\varphi \neq 0$ at $t = 0$) then

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (\partial_t u + \partial_x F(u))\varphi\,dx\,dt \\ &= -\int_0^\infty \int_{-\infty}^\infty u\partial_t\varphi\,dx\,dt - \int_{-\infty}^\infty \underbrace{u}_{=g}\varphi|_{t=0}\;dx - \int_0^\infty \int_{-\infty}^\infty F(u)\partial_x\varphi\,dx\;dt. \end{aligned}$$

Definition 89
$u \in L^\infty(\mathbb{R} \times (0, \infty))$ is called an integral solution of (16.1), (16.2) if

$$\int_0^\infty \int_{-\infty}^\infty (u\partial_t\varphi + F(u)\partial_x\varphi)\,dx\,dt + \int_{-\infty}^\infty g\varphi|_{t=0}\;dx = 0$$

holds for all $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty))$



If u is smooth except for a potential jump along the curve c through space-time, where ν is the normal, see Figure 16.1, and the support of $\varphi \Subset V = V_l \cup V_r$, then with $\nabla = (\partial_x, \partial_t)$

$$\begin{aligned}
0 &= \int_V (u\partial_t\varphi + F(u)\partial_x\varphi)\, dt\, dx \\
&= \int_{V_l} (F(u), u) \cdot \nabla\varphi\, dt\, dx + \int_{V_r} (F(u), u) \cdot \nabla\varphi\, dt\, dx \\
&= \int_c \varphi(F(u_l), u_l) \cdot \nu\, d\tau - \int_{V_l} \underbrace{(\partial_t u + \partial_x F(u))}_{=0\ u \text{ classical sol}} \varphi\, dt\, dx \\
&\qquad + \int_c \varphi(F(u_r), u_r) \cdot (-\nu)\, d\tau - \int_{V_r} \underbrace{(\partial_t u + \partial_x F(u))}_{=0\ u \text{ classical sol}} \varphi\, dt\, dx \\
&= \int_c \varphi(\nu_1(F(u_l) - F(u_r)) + \nu_2(u_l - u_r)\, d\tau
\end{aligned}$$

where $u_l(y) = \lim_{V_l \ni x \to y} u(x)$ and $u_r(y) = \lim_{V_r \ni x \to y} u(x)$ for $y \in c$ are the left and right sided limits of the jump.

Since this has to holds for all φ we get

$$(F(u_l) - F(u_r))\nu_1 + (u_l - u_r)\nu_2 = 0$$

along c. Note that when c is parametrized by t, i.e. $c = \left\{(x, t) \in \mathbb{R}^2 \,\middle|\, x = s(t)\right\}$, then

$$\nu(t) = \frac{1}{\sqrt{1 + s'^2}}(1, -s')$$

and therefore

$$F(u_l) - F(u_r) = s'(u_l - u_r)$$

along the curve.

16.2 Rankine-Hugoniot Condition

Theorem 90 (Rankine-Hugoniot Condition)
The condition

$$[[F(u)]] = \sigma[[u]]$$

where

$$\begin{aligned} [[F(u)]] &= F(u_r) - F(u_l) && \text{is the jump in } F(u) \\ [[u]] &= u_r - u_l && \text{is the jump in } u \\ \sigma &= s' && \text{is the speed of the curve} \end{aligned}$$

has to be satisfied for all integral solutions.

Example 91
The Burgers' equation

$$\begin{aligned} u_t + uu_x &= 0 && \text{in } \mathbb{R} \times (0, \infty) \\ u &= g && \text{on } \mathbb{R} \times \{0, \infty\} \end{aligned}$$

is a conservation law with $\partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = 0$ and quasilinear. Therefore it's characteristics are given by

$$\begin{aligned} \bar{x}' &\overset{(13.10)}{=} b_1 = u(\bar{x}) = \bar{z} \\ \bar{t}' &\overset{(13.10)}{=} b_2 = 1 \\ \bar{z}' &\overset{(13.11)}{=} 0 \end{aligned}$$

and therefore $\bar{z}$ is constant, i.e. u is constant along the curves

$$(\bar{x}(s), \bar{t}(s)) = (u(\bar{x}(s)), s),$$

and if the curve goes through (x, t) at $s = s_1$, then

$$u(x, t) = u(\bar{x}(s_1), \bar{t}(s_1)) = \bar{z}(s_1) = \bar{z}(0) = g(x_0).$$

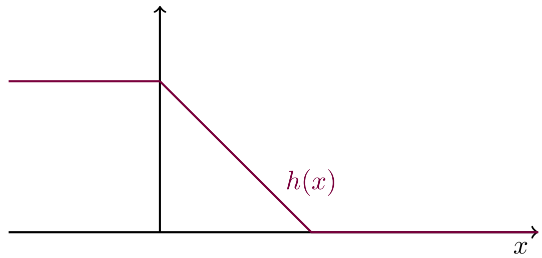
So the (projected) characteristics are tangent to $(u, 1)$ and u is constant along them.

Lecture 19 (March 18)

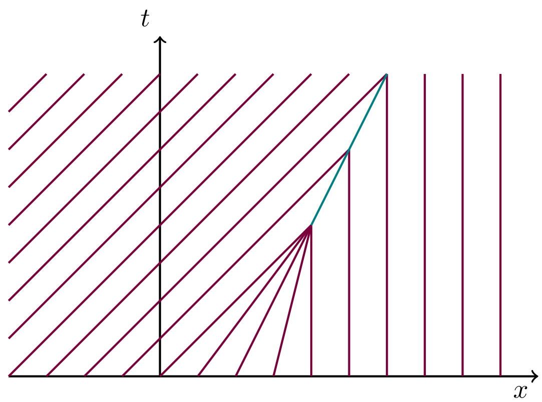
1. Assume now explicitly that

$$g(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x & 0 \leq x \leq 1 \\ 0 & 1 \leq x \end{cases}$$

then



and the projected characteristics look like



So u is constant along the maroon lines but what happens at the teal line when the characteristics cross?

The solution is given by

$$u(x,t) = \begin{cases} 1 & x \le t \\ \frac{1-x}{1-t} & t \le x \le 1 \\ 0 & 1 \le x \end{cases}$$

for $t \le 1$, i.e. when the characteristics did not cross yet.

From the picture we assume $s(t) = \frac{1+t}{2}$ and

$$u(x,t) = \begin{cases} 1 & x < s(t) \\ 0 & s(t) < x \end{cases}$$

and check the Rankine-Hugoniot condition

$$\begin{aligned} \sigma = s' &&= \frac{1}{2}, \\ [[F(u)]] = \frac{1}{2}u_r^2 - \frac{1}{2}u_l^2 &&= \frac{1}{2}, \\ [[u]] = u_r - u_l &&= 1 \end{aligned}$$

and therefore

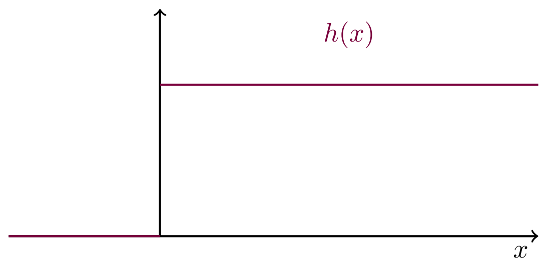
$$[[F(u)]] = \frac{1}{2} = \sigma[[u]]$$

and it is satisfied.

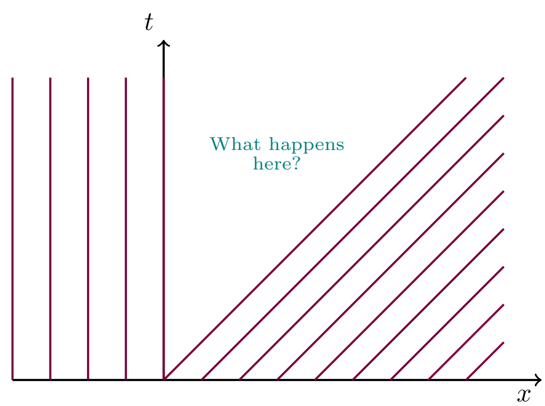
We have shock formation!

2. Assume now the opposite problem

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x \end{cases}$$

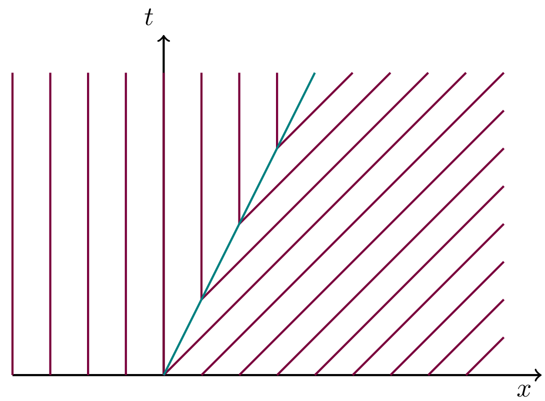


Then the characteristics are



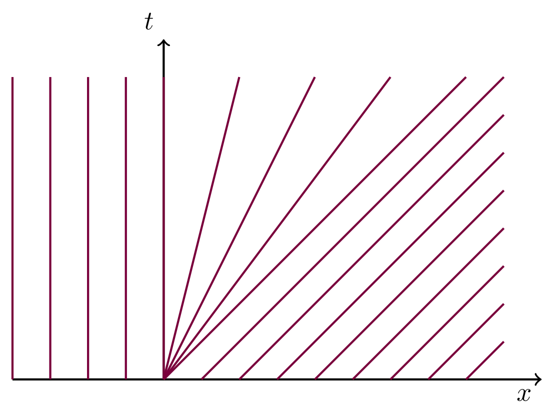
We can set

$$u_1(x,t) = \begin{cases} 0 & x < \frac{t}{2} \\ 1 & \frac{t}{2} < x \end{cases}$$



which is constant and satisfies the Rankine-Hugoniot condition at the jump and is therefore an integral solution. Or we can set

$$u_2(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < t \\ 1 & t < x \end{cases}$$



which is continuous so trivially satisfies the Rankine-Hugoniot condition and $u_t + uu_x = -\frac{x}{t^2} + \frac{x}{t}\frac{1}{t} = 0$ for $0 < x < t$.

Both are integral solutions of Burgers' equation! So this class is not unique!

How to select a physical solution? The idea is that moving backwards in time the characteristics should never cross. Then entropy will increase in time.

This yields the following.

16.3 Entropy Condition

Remark 92 (Entropy Condition)
The entropy condition states that at a discontinuity

$$F'(u_l) > \sigma > F'(u_r)$$

u_2 is continuous so satisfies the entropy condition but u_1 does not since

$$F'(u_l) = u_l = 0 < \frac{1}{2} = s' = \sigma < 1 = u_r = F'(u_r).$$

Strictly speaking for uniqueness (Evans 2010, Chapter 3.4, Theorem 3) one needs a bit more: F needs to be uniformly convex (which it is in Burgers' equation) and a more refined entropy condition (Evans 2010, Chapter 3.4, p. 149 (ii)).

Chapter E

Second Order Elliptic Equations

We will consider $U \subset \mathbb{R}^n$ open with sufficiently smooth boundary ∂U and

$$Lu = f \qquad \text{in } U \tag{16.1}$$
$$u = 0 \qquad \text{on } \partial U \tag{16.2}$$

where $f : U \to \mathbb{R}$ and

$$Lu = -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u \tag{16.3}$$

are given and $u : \overline{U} \to \mathbb{R}$.

Non homogeneous boundary conditions will be discussed shortly in Remark 105

Definition 93 (Ellipticity)
L is called uniformly elliptic if there exists $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{16.4}$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$. This means that $A(x)$ the matrix with elements a_{ij} is positive definite for a.e. x and smallest eigenvalue greater or equal to θ.

From now on we will assume L is uniformely elliptic.

Example 94
For $a_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$, $b = 0$, $c = 0$ then $L = \Delta$ and we get the Laplace equation.

17 Weak solutions

Testing (16.1) with $\varphi \in C_c^\infty(U)$, i.e. multiplying and integrating by parts and the boundary condition (16.2), yields

$$\begin{aligned}
\int_U f\varphi\,dx &= \int_U Lu\varphi\,dx \\
&= -\sum_{i,j=1}^n \int_U \partial_i \cdot (a_{ij}\partial_j u)\varphi\,dx + \int b\cdot\nabla u\varphi\,dx + \int_U cu\varphi\,dx \\
&= -\sum_{i,j=1}^n \int_{\partial U} \nu_i \cdot (a_{ij}\partial_j u)\varphi\,dx + \sum_{i,j=1}^n \int_U a_{ij}\partial_j u\partial_i\varphi\,dx + \int b\cdot\nabla u\varphi\,dx \\
&\quad + \int_U cu\varphi\,dx \\
&= \sum_{i,j=1}^n \int_U a_{ij}\partial_j u\partial_i\varphi\,dx + \int b\cdot\nabla u\varphi\,dx + \int_U cu\varphi\,dx \\
&=: B[u,\varphi]
\end{aligned}$$

Definition 95

1. The Bilinear form $B[\ ,\ ]$ associated with L given by (16.3) is

$$B[u,v] = \sum_{i,j=1}^n \int_U a_{ij}\partial_j u\partial_i v\,dx + \int_U b\cdot\nabla uv\,dx + \int_U cuv\,dx$$

   for all $u, v \in H_0^1(U)$.

2. $u \in H_0^1(U)$ is called a weak solution of (16.1), (16.2) for $f \in L^2$ if

$$B[u,v] = \langle f, v\rangle_{L^2}$$

   for all $v \in H_0^1$.

3. More generally $u \in H_0^1(U)$ is called a weak solution of (16.1), (16.2) for $f \in H^{-1}$ if

$$B[u,v] = \langle f, v\rangle_{H^{-1}\times H_0^1}$$

   for all $v \in H_0^1$. Remember $H^{-1} = (H_0^1)^\star$

Definition 96 (Classical, Strong, Weak, Very Weak solution)
Although the terminology varies depending on the author usually (disregarding boundary conditions)

- u is called a classical solution of $Lu = f$ if $u \in C^2$ and

$$Lu = f$$

holds pointwise.

- u is called a strong solution of $Lu = f$ if $u \in H^2$

$$Lu = f$$

holds almost everywhere.

- u is called a weak solution of $Lu = f$ if $u \in H^1$

$$B[u, v] = \langle f, v \rangle_{H^{-1} \times H_0^1}$$

for all $v \in H_0^1$, i.e.

$$Lu = f$$

in H^{-1}

- u is called a very weak solution of $Lu = f$ if $u \in L^2$

$$Lu = f$$

in H^{-2}

- Similar for a m-th order partial differential operator
  - Classical $u \in C^m$ and $Lu = f$ in $\mathcal{D}^\star \implies$ pointwise
  - Strong $u \in W^{m,p}$ and $Lu = f$ in $\mathcal{D}^\star \implies$ a.e.
  - Weak $u \in W^{\frac{m}{2},p}$ and $Lu = f$ in $\mathcal{D}^\star$
  - Very Weak $u \in L^p$ and $Lu = f$ in $\mathcal{D}^\star$

Lecture 20 (March 23)

18 Existence of solutions

Theorem 97 (Lax-Milgram)
Assume that H is a real Hilbert space and

$$B : H \times H \to \mathbb{R}$$

is bilinear and there exists $\alpha, \beta > 0$ such that

$$|B[u, v]| \leq \alpha \|u\|_H \|v\|_H$$

for all $u, v \in H$ and

$$B[u,u] \geq \beta \|u\|_H^2 \tag{18.1}$$

for all $u \in H$. Let f be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that

$$B[u,v] = \langle f, v \rangle_{H^\star \times H}$$

for all $v \in H$.

Definition 98
(18.1) is called coercivity condition.

Proof

1. For fixed $u \in H$ the mapping $v \mapsto B[u,v]$ is bounded linear functional on H and therefore Riesz representation theorem yields the existence of a unique $w \in H$ such that

   $$B[u,v] = (w,v)_H$$

   where $(\cdot,\cdot)_H$ is the inner product on H for all $v \in H$. We define the operator that maps $u \to w$ as A and can write

   $$B[u,v] = (A(u),v)_H \tag{18.2}$$

2. $A : H \to H$ is bounded and linear since

   $$\|A(u)\|_H^2 = (A(u), A(u))_H = B[u, A(u)] \leq \alpha \|u\|_H \|A(u)\|_H$$

   and for all $u_1, u_2, v \in H$ and $\lambda_1, \lambda_2 \in \mathbb{R}$

   $$\begin{aligned}(A(\lambda_1 u_1 + \lambda_2 u_2), v)_H &= B[\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 (A(u_1), v)_H + \lambda_2 (A(u_2), v)_H \\ &= (\lambda_1 A(u_1) + \lambda_2 A(u_2), v)_H\end{aligned}$$

   So write $A(u) = Au$ from now on.

3. We have

   $$\beta \|u\|_H^2 \leq B[u,u] = (Au, u)_H \leq \|Au\|_H \|u\|_H$$

   and therefore $\beta \|u\|_H \leq \|Au\|_H$ implying

   - A is one-to-one
   - $R(a) = \text{range}(A)$ is closed in H (see Thrm 1 here for example)

4. Actually
$$R(A) = H \tag{18.3}$$
Suppose not, then since $H = R(A) \oplus R(A)^\perp$ and $R(A)$ is closed, $R(A)^\perp$ is open and so there exists $0 \neq w \in \mathbb{R}(A)^\perp$, which contradicts
$$\beta \|w\|_H^2 \leq B[w,w] = (Aw, w) = 0.$$

5. Again by the Riesz Representation since f is a bounded linear functional on H, there exists $\tilde{w} \in H$ such that
$$\langle f, v\rangle_{H^\star \times H} = (\tilde{w}, v)_H \overset{(18.3)}{=} (A\tilde{u}, v)_H \overset{(18.2)}{=} B[\tilde{u}, v]$$

6. This $\tilde{u}$ is unique, since for $B[\bar{u}, v] = \langle f, v\rangle_{H^\star \times H}$ and $B[\tilde{u}, v] = \langle f, v\rangle_{H^\star \times H}$ by linearity of B
$$B[\tilde{u} - \bar{u}, v] = B[\tilde{u}, v] - B[\bar{u}, v] = 0$$
for all $v \in H$ and therefore
$$\beta \|\tilde{u} - \bar{u}\|_H^2 \leq B[\tilde{u} - \bar{u}, \tilde{u} - \bar{u}] = 0.$$

□

Remark 99
If B is symmetric, i.e.
$$B[u,v] = B[v,u]$$
for all $u, v \in H$ then it defines an inner product on H equivalent to its norm and therefore Riesz Representation Theorem yields the result directly.

Corollary 100 (Existence of Weak Solutions)
Let $a_{ij}, c \in L^\infty(U)$, $Lu = -\sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j u) + c(x)u$ with $c \geq 0$ be uniformly elliptic, i.e. $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$, $f \in H^{-1}$ then there exists a unique weak solution $u \in H^1$ to
$$\begin{aligned} Lu &= f \qquad \text{in } U \\ u &= 0 \qquad \text{on } \partial U \end{aligned}$$

Proof
The form
$$\begin{aligned} B &: H_0^1 \times H_0^1 \to \mathbb{R} \\ B[u,v] &= \sum_{i,j=1}^n \int_U a_{ij}\partial_j u \partial_i v \, dx + \int_U cuv \, dx \end{aligned}$$

is bilinear, continuous since by Hölder's inequality

$$\begin{aligned}|B[u,v]| &\leq \|a\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \|c\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} \\ &\leq \underbrace{(\|a\|_{L^\infty} + \|c\|_{L^\infty})}_{=:\alpha}\|u\|_{H^1}\|v\|_{H^1}\end{aligned}$$

and by Poincaré inequality

$$\|u\|_{L^2} \leq \tilde{c}\|\nabla u\|_{L^2}$$

implying

$$\begin{aligned}\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 &\leq (1+\tilde{c})\int_U \nabla u \cdot \nabla u \, dx \\ &\leq (1+\tilde{c})\int_U \sum_{i,j=1}^n \alpha_{ij}\partial_i u \cdot \partial_j u \, dx \\ &\leq (1+\tilde{c})\left[\sum_{i,j=1}^n \int_U a_{ij}\partial_j u \partial_i u \, dx + \int_U cu^2 \, dx\right] \\ &= \underbrace{(1+\tilde{c})}_{=:\frac{1}{\beta}} B[u,u]\end{aligned}$$

for all $u \in H_0^1$. So Lax-Milgram 97 yields the claim. □

Corollary 101
For uniformely elliptic

$$Lu = -\sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^n b_i(x)\partial_i u + c(x)u$$

with $a_{ij}, b_i, c \in L^\infty(U)$ there exists $\gamma \geq 0$ such that for every $\mu \geq \gamma$ and $f \in H^{-1}$ there exists a solution $u \in H^1$ of

$$\begin{aligned} Lu + \mu u &= f \qquad \text{in } U \\ u &= 0 \qquad \text{on } \partial U \end{aligned}$$

Sketch of Proof
The only problematic estimate proving the coercivity conditions

$$\beta\|u\|_{H^1}^2 \leq B[u,u].$$

- For $b = 0$ set $\tilde{L} = L + \|c\|_{L^\infty}$ such that $\tilde{c} \geq 0$, which is covered by Corollary 100. So $\gamma = \|c\|_{L^\infty}$ yields the claim.

- For $b \neq 0$ one can estimate the middle term by Hölder and Young's as follows

$$\left| \int_U b \cdot \nabla u u \, dx \right| \leq \|b\|_{L^\infty} \|\nabla u\|_{L^2} \|u\|_{L^2} \leq \varepsilon \|\nabla u\|_{L^2}^2 + \frac{1}{4\varepsilon} \|b\|_{L^\infty}^2 \|u\|_{L^2}^2 \tag{18.4}$$

for every $\varepsilon > 0$. Choosing ε sufficiently small this term can be compensated by the ellipticity constant and the $\frac{1}{4\varepsilon}\|b\|_{L^\infty}$ will be absorbed by γ as

$$\begin{aligned}
\|u\|_{H^1}^2 &= \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \\
&\overset{(18.4)}{\leq} \left(1 + \frac{2\varepsilon}{\theta}\right) \|\nabla u\|_{L^2}^2 + \left(1 + \frac{2}{4\varepsilon\theta}\|b\|_{L^\infty}^2\right) \|u\|_{L^2}^2 \\
&\quad - \frac{2}{\theta} \left| \int_U b \cdot \nabla u u \, dx \right| \\
&\overset{(16.4)}{\leq} \frac{1}{\theta}\left(1 + \frac{2\varepsilon}{\theta}\right) \int_U \sum_{i,j} a_{ij} \partial_i u \partial_j u \, dx + \frac{2}{\theta} \int_U b \cdot \nabla u u \, dx \\
&\quad + \frac{2}{\theta} \int c u^2 \, dx + \left(1 + \frac{2}{4\varepsilon\theta}\|b\|_{L^\infty}^2 + \frac{2}{\theta}\|c\|_{L^\infty}\right) \|u\|_{L^2}^2 \\
&\overset{\varepsilon = \frac{\theta}{2}}{\leq} \frac{2}{\theta} \left[\int_U \sum_{i,j} a_{ij} \partial_i u \partial_j u \, dx + \int_U b \cdot \nabla u u \, dx + \int c u^2 \, dx \right] \\
&\quad + \frac{2}{\theta} \underbrace{\left(\frac{\theta}{2} + \frac{1}{2\theta}\|b\|_{L^\infty}^2 + \|c\|_{L^\infty} \right)}_{=:\gamma} \|u\|_{L^2}^2 \\
&\leq \frac{2}{\theta} \left[B[u,u] + \mu \|u\|_{L^2}^2 \right] = \frac{2}{\theta} \tilde{B}[u,u],
\end{aligned}$$

where $\tilde{B}$ is the bilinear form associated to $\tilde{L}u = Lu + \mu u$.

This choice of γ is most likely not optimal!

□

Example 102
The boundary value problem for the Poisson equation

$$-\Delta u = f \qquad \text{in } U \tag{18.5}$$
$$u = 0 \qquad \text{on } \partial U \tag{18.6}$$

has a unique solution $u \in H^1$ for every $f \in H^{-1}$.

Lecture 21 (March 25)

19 Neumann Boundary Conditions

Consider

$$\begin{aligned} -\Delta u &= f && \text{in } U \\ \nu \cdot \nabla u &= 0 && \text{on } \partial U \end{aligned}$$

and $u \in C^2$. Then for all $v \in C^\infty(U)$

$$\begin{aligned} 0 &= \int_U (\Delta u + f) v \, dx \\ &= \int_{\partial U} \underbrace{n \cdot \nabla u}_{=0} v \, dS + \int_U \nabla u \cdot \nabla v \, dx - \int_U f v \, dx. \end{aligned}$$

Note that we need the compatibility condition

$$0 = \int_{\partial U} \nu \cdot \nabla u \, dS = \int_U \Delta u \, dx = \int_U f \, dx$$

so f has to be average free.

Definition 103
$u \in H^1(U)$ is called a weak solution of (18.5), (18.6) for $f \in L^2$ average free if

$$\int_U \nabla u \cdot \nabla v \, dx = \int_U f v \, dx$$

for all $v \in H^1(U) \supset H_0^1(U)$.

Corollary 104
For $U \subset \mathbb{R}^n$ with Lipschitz boundary and every average free $f \in L^2$ there exists a (up to additive constants) unique solution $u \in H^1$ of (18.5), (18.6).

Proof
We need to show

$$B[u, v] = \int_U \nabla u \cdot \nabla v \, dx$$

satisfies

$$|B[u, v]| \leq \|u\|_{H^1} \|v\|_{H^1}$$

which follows directly by Hölder's inequality and there exists a constant $\beta > 0$ such that

$$\beta \|u\|_{H^1}^2 = B[u, u]$$

For average free $u \in H^1$ this follows by the Poincaré inequality as

$$\begin{aligned}\|u\|_{H^1}^2 &= \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \le c\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = (c+1)\|\nabla u\|_{L^2}^2 \\ &= \underbrace{(c+1)}_{=:\frac{1}{\beta}} B[u,u].\end{aligned}$$

So by Lax-Milgram 97 there exists a unique solution u in the class $H^1 \cap \int_U u\,dx = 0$. The claim now follows since adding constants does not change the definition. □

Remark 105 (Non-Homogeneous Boundary Conditions)
If we want to solve

$$\begin{aligned} Lu &= f && \text{in } U \\ u &= g && \text{on } \partial U \end{aligned}$$

then every $g \in H^{\frac{1}{2}}(\partial U)$ (see wiki for definition of $H^{\frac{1}{2}}$) is the boundary (in the sense of Traces Evans 2010, Chapter 5.5) of an $H^1(U)$ function w (see wiki).

In essence, for $u \in H^1(U)$ the space $g \in H^{\frac{1}{2}}(\partial U)$ is exactly the space for which it makes sense to define $u|_{\partial U}$. And conversely for every $g \in H^{\frac{1}{2}}(\partial U)$ there exists a (non-unique) $w \in H^1(U)$ such that $w|_{\partial U} = g$.

Therefore defining $\tilde{u} = u - w$ it has the form

$$\begin{aligned} L\tilde{u} &= \tilde{f} := f - Lw && \text{in } U \\ \tilde{u} &= 0 && \text{on } \partial U \end{aligned}$$

and then $u = w + \tilde{u}$ so it suffices to solve the zero boundary condition problem. There are some caveats to this. In principle one can similarly solve the non-homogeneous Neumann boundary problem.

20 Regularity

In Theorems 24 and 27 we have seen that if u solves

$$\Delta u = 0 \qquad \text{in } U$$

then u is C^∞ and even analytic inside of U. What can one say about

$$\begin{aligned} \Delta u &= f && \text{in } U \\ u &= g && \text{on } \partial U \end{aligned}$$

or the Neumann boundary value problem?

Consider $U = \mathbb{R}^n$ or $U = \Pi^n$, the torus, i.e. $[0, L]^n$ with periodic boundary conditions, such that the boundary does not have an influence on the solution. Then if $u \in C^\infty(U)$ solves

$$\Delta u = f \qquad \text{in } U$$

one has by integration by parts

$$\begin{aligned}\|f\|_{L^2}^2 = \|\Delta u\|_{L^2}^2 &= \int_U \Delta u \Delta u \, dx = \underbrace{\int_{\partial U} \nu \Delta u \cdot \nabla u \, dx}_{=0} - \int_U \nabla \Delta u \cdot \nabla u \, dx \\ &= \int_U \nabla\nabla u \colon \nabla\nabla u \, dx - \underbrace{\int_{\partial U} \nu \nabla u \colon \nabla\nabla u \, dx}_{=0} = \|\nabla^2 u\|_{L^2}\end{aligned}$$

By approximation this holds for all $u \in H^2$.

Since ∇u is average free on U (otherwise $u \notin L^2(\mathbb{R}^n)$ or can not have periodic boundary conditions on Π^n) Poincaré inequality, i.e. Theorem 80, yields

$$\begin{aligned}\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 &\leq \|u\|_{L^2}^2 + (1 + c_{\mathrm{PC}})\|\nabla^2 u\|_{L^2}^2 \\ &= \|u\|_{L^2}^2 + (1 + c_{\mathrm{PC}})\|f\|_{L^2}^2 < \infty\end{aligned}$$

by assumption. Similarly $\nabla^k u$ solves

$$\Delta \nabla^k u = \nabla^k \Delta u = \nabla^k f$$

and therefore

$$\|u\|_{H^{k+2}}^2 \leq \|u\|_{L^2}^2 + c\|f\|_{H^k}^2$$

showing that u is automatically twice more regular than f.

Corollary 106

For $U \subset \mathbb{R}^n$ with C^2 boundary and every average free $f \in L^2$ there exists a (up to additive constants) unique solution $u \in H^2$ of

$$\begin{aligned}-\Delta u &= f && \text{in } U \\ \nu \cdot \nabla u &= 0 && \text{on } \partial U.\end{aligned}$$

Proof

By Corollary 104 there exists a unique solution $u \in H^1$. Similar to before assume $u \in C^\infty$, then

$$\begin{aligned}\|f\|_{L^2}^2 = \|\Delta u\|_{L^2}^2 &= \int_U \Delta u \, \Delta u \, dx = \int_{\partial U} \underbrace{\nu \cdot \nabla u}_{=0} \Delta u \, dx - \int_U \nabla u \cdot \nabla \Delta u \\ &= -\int_{\partial U} \nabla u \cdot (\nu \cdot \nabla) \nabla u \, dx + \|\nabla^2 u\|_{L^2}^2 \qquad (20.1)\end{aligned}$$

- Assume that the boundary is piecewise flat. Without loss of generality take $U = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$. Otherwise estimate similarly on the flat portions. Then

$$\nu = -e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \qquad \partial_n u = 0 \qquad \nabla u = \begin{pmatrix} \partial_1 u \\ \vdots \\ \partial_{n-1} u \\ 0 \end{pmatrix}$$

on ∂U and taking the tangential derivatives of the boundary condition

$$\partial_i \partial_n u = 0 \text{ for } i = 1, \dots, n-1$$

Therefore

$$\begin{aligned} \int_{\partial U} \nabla u \cdot (\nu \cdot \nabla) \nabla u \, dx &= -\sum_{i=1}^{n-1} \int_{\partial U} \partial_i u \partial_n \partial_i u \, dx \\ &= -\sum_{i=1}^{n-1} \int_{\partial U} \partial_i u \underbrace{\partial_i \partial_n u}_{=0} \, dx = 0 \end{aligned} \tag{20.2}$$

implying

$$\begin{aligned} \|u\|_{H^2}^2 &= \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \\ &\overset{(20.1)}{=} \|u\|_{H^1}^2 + \|f\|_{L^2}^2 + \int_{\partial U} \nabla u \cdot (\nu \cdot \nabla) \nabla u \, dx \\ &\overset{(20.2)}{=} \|u\|_{H^1}^2 + \|f\|_{L^2}^2 \end{aligned} \tag{20.3}$$

- If the boundary is not flat then similarly the only contribution to ∇u on ∂U is tangential, i.e.

$$\nabla u = \sum_{i=1}^{n-1} \tau (\tau \cdot \nabla) u \tag{20.4}$$

for a basis of tangent vectors τ_i. Since $\nu \cdot \nabla u = 0$ taking the tangential derivative we have

$$0 = \tau \cdot \nabla (\nu \cdot \nabla u) = \nabla u \cdot \underbrace{(\tau \cdot \nabla) \nu}_{\leq C(U)} + \tau \cdot (\nu \cdot \nabla) \nabla u. \tag{20.5}$$

Therefore the boundary term in (20.1) can be estimated by

$$\begin{aligned} \left| \int_{\partial U} \nabla u \cdot (\nu \cdot \nabla) \nabla u \, dx \right| &\overset{(20.4)}{=} \left| \int_{\partial U} (\tau \cdot \nabla) u \tau \cdot (\nu \cdot \nabla) \nabla u \, dx \right| \\ &\overset{(20.5)}{=} \left| \int_{\partial U} (\tau \cdot \nabla) u \nabla u \cdot \underbrace{(\tau \cdot \nabla) \nu}_{\leq \tilde{C}(U)} \, dx \right| \\ &\leq \tilde{C}(U) \int_{\partial U} |\nabla u|^2 \, dx \end{aligned} \tag{20.6}$$

Now extending ν defined on the boundary to ζ defined on U with $\zeta|_{\partial U} = \nu$ we find using divergence theorem and Young's inequality

$$\begin{aligned}\int_{\partial U} |\nabla u|^2 \, dx &= \int_{\partial U} \nu \cdot \zeta |\nabla u|^2 \, dx = \int_U \nabla \cdot (\zeta |\nabla u|^2) \, dx \\ &= \int_U \underbrace{\nabla \cdot \zeta}_{C(U)} |\nabla u|^2 \, dx + 2 \int_U \nabla u \cdot (\zeta \cdot \nabla) \nabla u \, dx \\ &\leq C(U) \|\nabla u\|_{L^2}^2 + 2 \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \\ &\leq C_{\tilde{\varepsilon}} \|\nabla u\|_{L^2}^2 + \tilde{\varepsilon} \|\nabla^2 u\|_{L^2}^2 \end{aligned} \tag{20.7}$$

for every $\tilde{\varepsilon} > 0$. Combining (20.1), (20.6), and (20.7) and setting $\varepsilon = \frac{\tilde{\varepsilon}}{\tilde{C}}$

$$\begin{aligned}\|u\|_{H^2}^2 &= \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \\ &= \|u\|_{H^1}^2 + \|f\|_{L^2}^2 + \int_{\partial U} \nabla u \cdot (\nu \cdot \nabla) \nabla u \, dx \\ &\leq \|u\|_{H^1}^2 + \|f\|_{L^2}^2 + \tilde{C} \int_{\partial U} |\nabla u|^2 \, dx \\ &\leq C_\varepsilon \|u\|_{H^1}^2 + \|f\|_{L^2}^2 + \varepsilon \|\nabla^2 u\|_{L^2}^2 \end{aligned}$$

for any $\varepsilon > 0$ and choosing $\varepsilon = \frac{1}{2}$ it can be absorbed such that

$$\|u\|_{H^2}^2 \leq 2\|f\|_{L^2}^2 + C\|u\|_{H^1}^2. \tag{20.8}$$

Now approximate $u \in H^1$ by these $u \in C^\infty$, then this holds for all $u \in H^1$, showing that $u \in H^2$ since the right-hand side of (20.3), and (20.8) is finite since by assumption $f \in L^2$ and by Corollary 104 $u \in H^1$. □

Using cut-off functions and approximating derivatives by difference quotients the interior regularity can be generalized to elliptic operators.

Proposition 107 (Interior Estimates; Evans 2010, Chapter 6.3.1, Theorems 1+2)
Assume $a_{ij}, b_i, c \in C^{k+1}(U)$ or merely $a_{ij} \in C^1(U)$, $b_i, c \in L^\infty(U)$ if $k = 0$ $f \in H^k(U)$ and $u \in H^1(U)$ is a weak solutions of

$$Lu = f \qquad \text{in } U.$$

Then $u \in H^{k+2}_{\text{loc}}(U)$ and

$$\|u\|_{H^{k+2}(V)} \leq c(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$$

for all $V \Subset U$ and $c(k, U, V, L) > 0$.

Remarks 108

1. This is a result just for the interior. Therefore we do not specify boundary conditions, but the result only holds away from the boundary, i.e. in $V \Subset U$ and equivalently H^{k+2}_{loc}.

2. In view of the Sobolev embedding, Theorem 76, if f is sufficiently regular, u will be a classical solution in the interior.

The general case with boundaries is more advanced.

In particular what what is the correct space of boundary data. See Remark 105, Evans 2010, Chapter 5.5 and wiki.

But the idea is similar to the one of Corollary 106. All tangential derivatives have to match the boundary condition and then estimate the normal part $\partial_n^2 u = \Delta u - \sum_{i=1}^{n-1} \partial_i^2 u = f - \sum_{i=1}^{n-1} \partial_i^2 u$ or $a_{nn}\partial_n^2 u = Lu - \dots$, where the right-hand side is now sufficiently regular.

Proposition 109 (Grisvard 1985, Theorem 2.5.1.1 + Corollary 2.5.2.2)
Let $k \in \mathbb{N}_0$, L uniformly elliptic, $b_i = 0$, $c \geq 0$ and $u \in H^2(U)$ solve

- $$\begin{aligned} Lu &= f && \text{in } U \\ u &= g && \text{on } \partial U \end{aligned}$$

  with $g \in W^{2+k-\frac{1}{p},p}(\partial U)$ or

- $$\begin{aligned} Lu &= f && \text{in } U \\ \nu \cdot \nabla u &= g && \text{on } \partial U \end{aligned}$$

  with $g \in W^{1+k-\frac{1}{p},p}(\partial U)$

and $f \in W^{k,p}(U)$. Then $u \in W^{k+2,p}(U)$.

Remark 110
Loosely speaking, u is twice more differentiable than f and what one looses by evaluating the boundary data.

References

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