Math 3IA3
Introduction to Real Analysis

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Contents

Lectures

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Based on Bartle and Sherbert 2011.

Notation

Here we use the following convention

- $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$
- Subsets/Supersets
  - $A \subset B$ denotes that A is a subset of B, i.e. all elements of A are also in B.
  - $A \subsetneq B$ denotes that A is a proper/strict subset of B, i.e. all elements of A are also in B and there exist elements in B that are not in A.
  - We won't use $\subseteq$.

1 Introduction

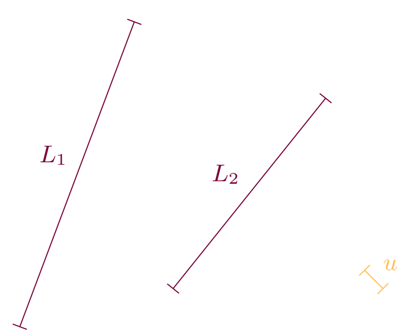
Lecture 1 (January 06)
We have to start somewhere, so the natural numbers are

$$\mathbb{N} = \{1, 2, 3, \dots\},$$

which are suitable for measuring discrete quantities. (We use this convention and write $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.) Obviously these are not enough. We can not measure continuous quantities like lengths, weights, ... Ancient Greeks would use comparisons to measure these quantities. For two lengths L_1 and L_2 they believed one can find a small enough unit u such that

$$L_1 = mu, \qquad L_2 = nu$$

with natural numbers m and n.



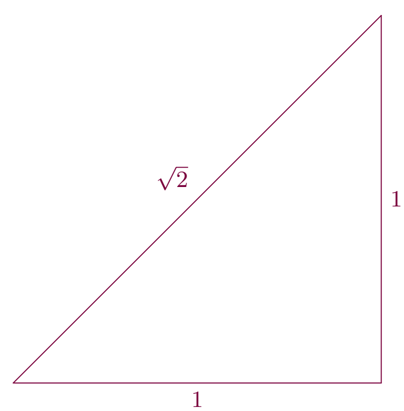
So

$$\frac{L_1}{L_2} = \frac{m}{n},$$

leading to the positive rational numbers

$$\mathbb{Q}^+ = \left\{ \frac{m}{n} \,\middle|\, m, n \in \mathbb{N} \right\}.$$

Around 500 BC, it was discovered that this is not always possible and the rational numbers are not enough. Measuring the diagonal of a square results in square roots



In mathematics we state properties, which might be inspired by observations or intuition, and then prove that these statements are in fact true.

Theorem 1
$\sqrt{2} \notin \mathbb{Q}^+$, i.e. $\sqrt{2}$ can not be written as a fraction of 2 natural numbers.

Accordingly we want to prove this theorem now. But how do we learn how to prove things? Some proofs are unintuitive and we have to learn them from others and memorize them and especially in the beginning it can be quite difficult to understand them. But over time we develop a better understanding of how and why these proofs work and it will become more easy.
This is why you should work together with friends and colleagues and come up with solutions/proofs together. The sometimes portrait stereotype of a mathematician working alone in his room on a problem for a long time and then coming up with a new result is the extreme exception. The vast majority of research articles are written in groups, and many single author publications are acknowledge discussions with other researchers.
But, in the tests, exams, ... you will be on your own. So I encourage you to work and study together but afterwards ask yourself if you would be able to

solve the problem on your own (maybe even try a closely related problem) or if you only think you understand what the others just told you.
And do not push this to the end it is much harder to catch up after some time and potentially you will not understand the new material if you do not understand old stuff.
Nevertheless, often times the statement of the theorem and our intuition, which will build up after some time, can guide us towards a proof. A good idea is to ask

- What do I know from the assumptions?
- What do I need to show?

Next we will go through an example proof.

Proof
We will prove Theorem 1 by contradiction. This is often the case for unintuitive proofs. This means we assume that $\sqrt{2} \in \mathbb{Q}^+$ and lead to a contradiction. What do we know by this assumption? By this assumption $\sqrt{2}$ can be written as

$$\sqrt{2} = \frac{m}{n}, \tag{1}$$

where $m, n \in \mathbb{N}$. Additionally we can assume that their only common factor is 1, as otherwise we could cancel the common factor. Squaring (1) yields

$$2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$$

and therefore

$$2n^2 = m^2 \tag{2}$$

Since $2n^2 = 2 \cdot \text{some natural number}$ we know that $2n^2$ is even. By (2) m^2 must be even. Since m^2 is even, m has to be even, too.[1] As m is even, we can write it as

$$m = 2k \tag{3}$$

for some $k \in \mathbb{N}$. Plugging (3) into (2) we get

$$2n^2 = m^2 = (2k)^2 = 4k^2$$

and after dividing by 2

$$n^2 = 2k^2.$$

Again, $2k^2$ is even and therefore n^2 is even, which implies[2] that n is even. We can therefore write it as

$$n = 2l \tag{4}$$

for some $l \in \mathbb{N}$. By (3) and (4), both m and n share the common factor 2, which contradicts the assumption that their only common factor is 1. □

[1] This has to be proven!
[2] Again this has to be proven!

The first proof of Theorem 1 probably did not look like this. Why would you assume that the fraction was irreducible in the beginning? Mathematics develops and proofs in textbooks are usually polished and elegant. In this course we will try to make these mistakes and fix them afterwards to show this process. So that you will be able to come up with strategies and proofs and be able to fix the first attempts.

Lecture 2 (January 08)

In the previous proof we skipped over the following theorem.

Theorem 2
Let $n \in \mathbb{N}$. If n^2 is even, then n is even.

Proof
We will show this by contraposition, i.e. we will show that if $n \in \mathbb{N}$ is odd, then n^2 is odd. Why do we do it this way? Going from n^2 to n is like taking a square root, which is much harder then squaring by going from n to n^2. If n is odd, we can write it as $n = 2k + 1$ for some $k \in \mathbb{N}_0$. We calculate

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \tag{5}$$

and see that $2(2k^2 + 2k) + 1 = 2 \cdot \text{some natural number} + 1$, so it must be odd. And by (5) n^2 is odd. □

Theorem 2 is acutally an equivalence.

Theorem 3
Let $n \in \mathbb{N}$. n^2 is even if and only if n is even.

Proof
We have to show

1. n^2 is even implies n is even, which we have done in Theorem 2.
2. n is even implies n^2 is even. (Do as an exercise) Since n is even, it can be written as $n = 2k$ for some $k \in \mathbb{N}$. Squaring the equation we get

   $$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

   Since $2(2k^2)$ is even n^2 has to be even.

□

2 Real numbers

2.1 The Algebraic Properties of $\mathbb{R}$

Field Axioms of $\mathbb{R}$
On $\mathbb{R}$, there are tow operations, denoted by $+$ and $\cdot$, called addition and multiplication, respectively. These operations satisfy

(A1) $a + b = b + a$ for all $a, b \in \mathbb{R}$ (commutative property of addition)

(A2) $(a+b)+c = a+(b+c)$ for all $a, b, c \in \mathbb{R}$ (associative property of addition)

(A3) there exists an element $0 \in \mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all $a \in \mathbb{R}$ (existence of a zero element)

(A4) for every $a \in \mathbb{R}$ there exists an element $-a \in \mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (existence of negative elements)

(M1) $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{R}$ (commutative property of multiplication)

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{R}$ (associative property of multiplication)

(M3) there exists an element $1 \in \mathbb{R}$ (distinct from 0) such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all $a \in \mathbb{R}$ (existence of a unit element)

(M4) for all $0 \neq a \in \mathbb{R}$ there exists an element $\frac{1}{a} \in \mathbb{R}$ such that $a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1$ (existence of reciprocals)

(D) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(b+c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in \mathbb{R}$ (distributive property of multiplication over addition)

Note that we did not introduce subtraction or division, yet. $-a$ and $\frac{1}{a}$ are just elements in $\mathbb{R}$. We have only written them like this because in the end subtraction will be addition by the negative elements and division will be multiplication with reciprocals.
With these Axioms all of the familiar algebraic properties can be derived. As this is a calculus course, and not a algebra course, we won't do it and only show a couple of important results.

Theorem 4

1. If $z, a \in \mathbb{R}$ and $z + a = a$, then $z = 0$.
2. If $u, b \in \mathbb{R}$, $b \neq 0$ and $u \cdot b = b$, then $u = 1$.
3. If $a \in \mathbb{R}$, then $a \cdot 0 = 0$

Proof

1. By (A3), (A4), (A2), the assumption $z + a = a$, and (A4) one has

$$z = z + 0 = z + (a + (-a)) = (z + a) + (-a) = a + (-a) = 0.$$

2. Using (M3),(M4),(M1),(M2),(M1), the assumption $u \cdot b = b$, and (M4) we get

$$u = 1 \cdot u = \left(b \cdot \frac{1}{b}\right) \cdot u = \left(\frac{1}{b} \cdot b\right) \cdot u = \frac{1}{b} \cdot (b \cdot u) = \frac{1}{b} \cdot (u \cdot b) = \frac{1}{b} \cdot b = 1.$$

3. By (M3), (D), (A3), (M3)

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1 + 0) = a \cdot 1 = a.$$

Therefore by 1. we have $a \cdot 0 = 0$.

□

Theorem 5

1. If $a, b \in \mathbb{R}$, $a \neq 0$ and $a \cdot b = 1$, then $b = \frac{1}{a}$
2. If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Proof

1. By (M1), (M4), (M2), the assumption $a \cdot b = 1$, (M3)

$$b = 1 \cdot b = \left(\frac{1}{a} \cdot a\right) \cdot b = \frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 1 = \frac{1}{a}$$

2. It suffices to assume that $a \neq 0$ and we show that $b = 0$ follows.*

Is it clear why?

- If $a = 0$ (or $b = 0$) we are already done since this is already the conclusion.
- We would also need to show that for $b \neq 0$ it follows that $a = 0$. But by (M1) and the assumption

$$b \cdot a = a \cdot b = 0$$

So we would need to show if $b \cdot a = 0$, $a, b \in \mathbb{R}$, $b \neq 0$ then $a = 0$. But this is exactly the same as * with a and b swapped. So it is covered if we show *.

On one hand by (M2), (M4), (M3)

$$\frac{1}{a} \cdot (a \cdot b) = \left(\frac{1}{a} \cdot a\right) \cdot b = 1 \cdot b = b. \tag{6}$$

On the other hand by the assumption $a \cdot b = 0$ and Theorem 4 Part 3

$$\frac{1}{a}(a \cdot b) = \frac{1}{a} \cdot 0 = 0 \tag{7}$$

Combining (7) and (6)

$$0 = \frac{1}{a}(a \cdot b) = b.$$

□

Theorem 5 Part 1 shows that the reciprocal (or multiplicative inverse) for non-zero elements is unique. Similar one can show that if $a + b = 0$, then $b = -a$, proving the uniqueness of the negative (or additive inverse). With this we can define the operations subtraction and division.

Definition 6
Let $a, b \in \mathbb{R}$.

- subtraction is defined by

$$a - b := a + (-b)$$

- If $b \neq 0$ division is defined by

$$\frac{a}{b} := a \cdot \frac{1}{b}$$

The sign := means the left side is defined by the right side.
So subtraction of b is just addition of the (unique) negative element $-b$. Similar division for $b = 0$.

Notation
In the following we will

- drop the $\cdot$ for multiplication, i.e. for $a, b \in \mathbb{R}$

$$ab = a \cdot b.$$

- use the standard notation for exponents, i.e. for $a \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\begin{aligned} a^n &= a^{n-1}a \\ a^1 &= a \\ a^0 &= 1 \text{ for } a \neq 0 \\ a^{-1} &= \frac{1}{a} \text{ for } a \neq 0 \\ a^{-n} &= \left(\frac{1}{a}\right)^n \text{ for } a \neq 0 \end{aligned}$$

- freely use the *standard techniques of algebra*

Lecture 3 (January 09)

Sets of Numbers

- The natural numbers
$$\mathbb{N} = \{1, 2, 3, \dots\}$$
- The integers
$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N},$$
where 0 is the zero element of $\mathbb{R}$ and $-\mathbb{N}$ consists of elements $-n$, which are identified as the n-fold sum of -1.
- The rational numbers
$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, b \neq 0 \right\}$$
- The irrational numbers are numbers in $\mathbb{R}$ that are not in $\mathbb{Q}$.
- The real numbers can be thought of as *all the standard numbers.* Every number in $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ is also in $\mathbb{R}$, i.e. $\mathbb{N}, \mathbb{Z}, \mathbb{Q} \subset \mathbb{R}$. $\pi, e, \sqrt{2}, \dots \in \mathbb{R}$. Note that ∞ and $-\infty$ are not real numbers.

Although this axiomatic approach to the real numbers might be a bit unsatisfactory, it has the advantage of skipping cumbersome constructions, such as construction via completion or Dedekind cuts.

Order Properties of $\mathbb{R}$ As before we axiomatically describe the following.

The Order Properties of $\mathbb{R}$
There exists a nonempty set $\mathbb{P} \subset \mathbb{R}$, called the positive real numbers, that satsifies the following properties

1. If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$
2. If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$
3. If $a \in \mathbb{R}$, then exactly one of the following holds
$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}$$

Part 3 is called the Trichotomy Property, which divides $\mathbb{R}$ into 3 distinct sets. It also allows us to define

- If $a \in \mathbb{P}$ we write $a > 0$ and say a is (strictly) positive
- If $-a \in \mathbb{P}$ we write $a < 0$ and say a is (strictly) negative
- If $a \in \mathbb{P} \cup \{0\}$ we write $a \geq 0$ and say a is nonnegative
- If $-a \in \mathbb{P} \cup \{0\}$ we write $a \leq 0$ and say a is nonpositive

With this we can define an ordering on $\mathbb{R}$

Definition 7
Let $a, b \in \mathbb{R}$.

1. If $a - b \in \mathbb{P}$ we write $a > b$ or $b < a$
2. If $a - b \in \mathbb{P} \cup \{0\}$ we write $a \geq b$ or $b \leq a$
3. For $c \in \mathbb{R}$ we write $a < b < c$ if $a < b$ and $b < c$. Similarly $>, \leq, \geq$.

Theorem 8
Let $a, b \in \mathbb{R}$. If $a \leq b$ and $b \leq a$, then $a = b$.

Proof
By the Trichotomy Property, only one of the following holds. $a-b \in \mathbb{P}$, $-(a-b) \in \mathbb{P}$ or $a - b = 0$.
Since $a \geq b$, by definition we have $a - b \in \mathbb{P} \cup \{0\}$. Since also $b \geq a$ we have $-(a-b) = b - a \in \mathbb{P} \cup \{0\}$, implying $a - b \notin \mathbb{P}$. As $a - b \in \mathbb{P} \cup \{0\}$ and $a - b \notin \mathbb{P}$ we get $a - b \in \{0\}$, so $a - b = 0$. $\square$

Similar one finds that either $a < b$, $b < a$ or $a = b$ for any $a, b \in \mathbb{R}$.

Theorem 9
Let $a, b, c \in \mathbb{R}$.

1. If $a > b$ and $b > c$, then $a > c$.
2. If $a > b$, then $a + c > b + c$.
3. (a) If $a > b$ and $c > 0$, then $ca > cb$
   (b) If $a > b$ and $c < 0$, then $ca < cb$

Proof

1. Since $a - b \in \mathbb{P}$ and $b - c \in \mathbb{P}$, Order Property 1 implies $a - c = (a - b) + (b - c) \in \mathbb{P}$
2. We have $(a + c) - (b + c) = a - b \in \mathbb{P}$, implying $a + c > b + c$.

3. (a) By assumption $c, a-b \in$ Pr. Therefore Order Property 2 implies $ca - cb = c(a-b) \in \mathbb{P}$ and therefore $ca > cb$
   (b) By assumption $-c, a-b \in$ Pr. Therefore Order Property 2 implies $cb - ca = (-c)(a-b) \in \mathbb{P}$ and therefore $cb > ca$.

□

Theorem 10

1. If $0 \neq a \in \mathbb{R}$, then $a^2 > 0$.
2. $1 > 0$.
3. If $n \in \mathbb{N}$, then $n > 0$.

Proof

1. By the Trichotomy Property, since $a \neq 0$, either $a \in \mathbb{P}$ or $-a \in \mathbb{P}$. If $a \in \mathbb{P}$, then by the Order Property 2 $a^2 = aa \in \mathbb{P}$. Similarly if $-a \in \mathbb{P}$, then by the Order Property 2 $a^2 = aa = (-a)(-a) \in \mathbb{P}$. Either way $a^2 \in \mathbb{P}$, implying $a^2 > 0$.
2. Since $1 = 1^2$ Part 1 implies $1 > 0$
3. We use mathematical induction.
   - Base Case: For $n = 1$, $n = 1 > 0$ was proven in Part 2
   - Induction Assumption: Suppose $n > 0$ holds for some $n \in \mathbb{N}$.
   - Induction Conclusion: For $n + 1$ we have $n \in \mathbb{P}$ by the induction assumption and $1 \in \mathbb{P}$ by Part 2. Therefore the Order Property 1 implies $n + 1 \in \mathbb{P}$, i.e. $n + 1 > 0$.

□

Lecture 4 (January 13)

Theorem 11
If $a \in \mathbb{R}$ fulfills $0 \leq a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

This theorem says that there exists no smallest positive number.

Proof
We want to show that can not be true since we could always find a smaller number by dividing that number by 2. So we first have to show that $\frac{1}{2} > 0$
By definition $\frac{1}{2}2 = 1$ and by Theorem 10.3 $1, 2 > 0$. By the Trichotomy Property $\frac{1}{2}$ is either positive, negative or 0. Since it is not 0, it can only be positive or

negative. If $\frac{1}{2}$ would be negative then $-1 = \left(-\frac{1}{2}\right) 2 \in \mathbb{P}$ by Order Property 2, which contradicts that $1 > 0$. So $\frac{1}{2}$ is positive.
With this we can prove the actual statement by contradiction.
Suppose there exists $a \in \mathbb{R}$ with $0 < a < \varepsilon$ for every $\varepsilon > 0$. Then setting $\varepsilon_0 = \frac{1}{2}a$, we find

$$\varepsilon_0 = \frac{1}{2}a > 0 \tag{8}$$

since $\frac{1}{2} > 0$ and $a > 0$. Using (8) twice we find

$$0 < \varepsilon_0 < \varepsilon_0 + \varepsilon_0 = 2\varepsilon_0 = 2\frac{1}{2}a = a,$$

which contradicts the assumption. □

We have seen that the product of two positive numbers is positive. The reverse is not necessarily true.

Theorem 12
Let $a, b \in \mathbb{R}$. If $ab > 0$, then either

- $a > 0$ and $b > 0$ or
- $a < 0$ and $b < 0$

Proof
By Theorem 4.3 and Axiom (M1) $a, b \neq 0$. So either $a > 0$ or $a < 0$. If $a > 0$, then $\frac{1}{a} > 0$ (analogously to the proof of Theorem 11) and therefore $b = \frac{1}{a}(ab) > 0$. The proof for $a < 0$ works in exactly the same way. □

Corollary 13
Let $a, b \in \mathbb{R}$. If $ab < 0$, then either

- $a < 0$ and $b > 0$ or
- $a > 0$ and $b < 0$

Proof
Since $ab < 0$ we have $(-a)b = -ab > 0$. Theorem 12 now shows either $-a, b > 0$ or $-a, b < 0$, so $a < 0$ and $b > 0$ or $a > 0$ and $b < 0$. □

Inequalities Here we want to see how to apply these rules to solve inequalities.

Example 14

- Determine the set A of all real numbers x such that $2x + 3 \leq 6$.
  - By Theorem 9

$$A = \{x \in \mathbb{R} \mid 2x + 3 \leq 6\} = \{x \in \mathbb{R} \mid 2x \leq 3\} = \left\{x \in \mathbb{R} \,\middle|\, x \leq \frac{3}{2}\right\}$$

- Determine the set $B = \left\{x \in \mathbb{R} \,\middle|\, x^2 + x > 2\right\}$.
  - $x \in B$ is equivalent to $x \in \mathbb{R}$ and

    $$x^2 + x > 2 \Leftrightarrow x^2 + x - 2 > 0 \Leftrightarrow (x-1)(x+2) > 0,$$

    which by Theorem 12 is equivalent to either $x - 1, x + 2 > 0$ or $x - 1, x + 2 < 0$.
    The first case corresponds to $x > 1$ and $x > -2$. Since if $x > 1$ trivially $x > -2$ the first case is equivalent to $x > 1$.
    Similar for the second case

    $$x - 1 < 0, x + 2 < 0 \Leftrightarrow x < 1, x < -2 \Leftrightarrow x < -2$$

    Combining the cases either $x > 1$ or $x < -2$, so

    $$B = \{x \in \mathbb{R} \,|\, x < -2\} \cup \{x \in \mathbb{R} \,|\, x > 1\}$$

- Determine the set $C = \left\{x \in \mathbb{R} \,\middle|\, \frac{2x+1}{x+2} < 1\right\}$
  - $x \in C$ is equivalent to $x \in \mathbb{R}$ and

    $$\frac{2x+1}{x+2} < 1 \Leftrightarrow \frac{2x+1}{x+2} - 1 < 0 \Leftrightarrow \frac{x-1}{x+2} < 0$$

    and therefore by Corollary 13 (and similar to Quiz 1.2 $a > 0 \Leftrightarrow \frac{1}{a} > 0$) either $x - 1 < 0$ and $x + 2 > 0$, which corresponds to $-2 < x < 1$, or $x - 1 > 0$ and $x + 2 < 0$, which corresponds to $x > 1$ and $x < -2$, which is impossible. Therefore $C = \{x \in \mathbb{R} \,|\, -2 < x < 1\}$.

Theorem 15
For $0 \le a, b \in \mathbb{R}$

$$a < b \Leftrightarrow a^2 < b^2$$

Proof
Without loss of generality assume that either a or b is non-zero. (Otherwise the statement is $0 < 0 \Leftrightarrow 0^2 < 0^2$, which both can not be satisfied)
If $b > a$ then $a^2 = aa \le ab \le bb = b^2$ by Theorem 9.
Conversely if $b^2 > a^2$ one has

$$0 < b^2 - a^2 = (b+a)(b-a)$$

and dividing by $b + a > 0$ (By Quiz 1.2 $a + b > 0$ implies $\frac{1}{a+b} > 0$. Therefore Theorem 9 shows that this leaves the inequality unchanged) shows $0 < b - a$. □

2.2 Absolute Value and the Real Line

Definition 16 (Absolute Value)
The absolute value (function) is defined by

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

The trichotomy property ensures that it is well defined for all real numbers. Note that it always returns the non-negative part of a.

Example 17

- $|3| = 3$
- $|-9| = 9$
- $|0| = 0$
- $|-a| = |a|$

Theorem 18
Let $a, b \in \mathbb{R}$.

1. $|ab| = |a|\,|b|$
2. $|a^2| = |a|^2$
3. Assume $b \geq 0$. Then $|a| \leq b$ if and only if $-b \leq a \leq b$.
4. $-|a| \leq a \leq |a|$.

Lecture 5 (January 15)

Proof

1. Assume $a, b \neq 0$. (For $a = 0$ trivially $|ab| = |0| = 0 = 0|b| = |a||b|$ and $b = 0$ similar)
    - If $a, b > 0$ one has $|ab| = ab = |a|\,|b|$
    - If $a > 0$, $b < 0$ one has $|ab| = -ab = a(-b) = |a|\,|b|$
    - $a < 0$, $b > 0$ similar to $a > 0$, $b < 0$
    - If $a, b < 0$ one has $|ab| = ab = (-a)(-b) = |a|\,|b|$
2. This follows directly from Part 1 by choosing $b = a$.

3.
   - Assume $a \geq 0$. If $|a| \leq b$ one has $-b \leq 0 \leq a = |a| \leq b$ and for $-b \leq a \leq b$ one has $|a| = a \leq b$.
   - Assume $a \leq 0$. If $|a| \leq b$ one has $-b \leq 0 \leq -a$ and $-a = |a| \leq b$, implying $a < b$ and $-b \leq a$. Conversely since $a = -(-a) = -|a|$ if $-b \leq a \leq b$ one has $-b \leq a = -|a|$, which implies $|a| \leq b$.

4. This follows directly from Part 3 by choosing $b = |a|$.

□

Theorem 19 (Triangle Inequality)
If $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Proof
By Part 4 of Theorem 18

$$-|a| \leq a \leq |a| \qquad \text{and} \qquad -|b| \leq b \leq |b|$$

Adding both left sides and right sides

$$-(|a| + |b|) \leq (a + b) \leq |a| + |b|,$$

which by Part 3 of Theorem 18 yields $|a + b| \leq |a| + |b|$. □

Corollary 20
If $a, b \in \mathbb{R}$, then

1. $\big||a| - |b|\big| \leq |a - b|$ (called reverse triangle inequality)
2. $|a - b| \leq |a| + |b|$

Proof

1. The triangle inequality yields

$$\begin{aligned} |a| = |a - b + b| \leq |a - b| + |b| &\implies |a| - |b| \leq |a - b| \\ |b| = |b - a + a| \leq |b - a| + |a| &\implies -|a - b| \leq |a| - |b| \end{aligned}$$

   or equivalently $-|a - b| \leq |a| - |b| \leq |a - b|$, which by Part 3 of Theorem 18 yields $\big||a| - |b|\big| \leq |a - b|$.

2. By the triangle inequality

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|.$$

□

Example 21

- Determine the set $A = \{x \in \mathbb{R} \mid \; |2x+3| \leq 7\}$.

  By Part 3 of Theorem 18

$$\begin{aligned} |2x+3| \leq 7 \quad &\Leftrightarrow \quad -7 \leq 2x+3 \leq 7 \qquad \Leftrightarrow \quad -10 \leq 2x \leq 4 \\ &\Leftrightarrow \quad -5 \leq x \leq 2 \end{aligned}$$

  so $A = \{x \in \mathbb{R} \mid -5 \leq x \leq 2\}$

- Determine $B = \{x \in \mathbb{R} \mid \; |x-1| \leq |x|\}$.

  - Method 1: Consider the cases where the absolute value changes behavior, i.e. at $x = 1$, $x = 0$. We distinguish 3 cases

    1. $x < 0$:

    $$|x-1| \leq |x| \quad \Leftrightarrow \quad -(x-1) \leq -x \quad \Leftrightarrow \quad 1 \leq 0 \quad \Leftrightarrow \quad \lightning$$

       So no $x < 0$ fulfills the condition

    2. $0 \leq x \leq 1$:

    $$|x-1| \leq |x| \quad \Leftrightarrow \quad -(x-1) \leq x \quad \Leftrightarrow \quad 1 \leq 2x \quad \Leftrightarrow \quad \frac{1}{2} \leq x$$

       which yields $\frac{1}{2} \leq x = 1$

    3. $1 \leq x$

    $$|x-1| \leq |x| \quad \Leftrightarrow \quad x-1 \leq x \quad \Leftrightarrow \quad 0 \leq 1 \quad \Leftrightarrow \quad \checkmark$$

       So all $x \geq 1$ fulfill the condition.

    Combined this shows that $B = \left\{x \in \mathbb{R} \,\middle|\, x \geq \frac{1}{2}\right\}$

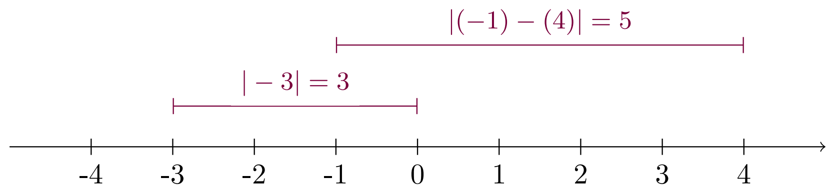
  - Method 2: Similar to Theorem 15 for $a, b \geq 0$

$$a \leq b \quad \Leftrightarrow \quad a^2 \leq b^2.$$

    Therefore

$$\begin{aligned} |x-1| \leq |x| \quad &\Leftrightarrow \quad x^2 - 2x + 1 = (x-1)^2 = |x-1|^2 \leq |x|^2 = x^2 \\ &\Leftrightarrow \quad -2x + 1 \leq 0 \quad \Leftrightarrow \quad x \geq \frac{1}{2}. \end{aligned}$$

We could also draw a sketch to get an intuition and then try to prove that this guess is actually correct.

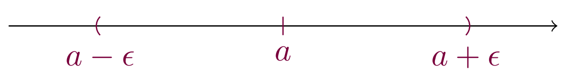


The Real Line The real line is a nice way of visualizing the real numbers. The absolute value function $|\cdot|$ corresponds to a distance function. $|a|$ is the distance of a to the origin and more generally the distance between a and b is $|a - b|$. More generally speaking (not important for this course) the absolute value function describes a metric on the real numbers, which together forms a metric space.
Later we will often investigate how close numbers are to each other, which we will do by using this distance function.

Definition 22 (ε-neighborhood)
Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε-neighborhood of a is the set $V_\varepsilon(a) := \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$

This means that the ε-neighborhood of a consists of all the points that are closer to a then ε

$$x \in V_\varepsilon(a) \quad \Leftrightarrow \quad |x - a| < \varepsilon \quad \Leftrightarrow \quad a - \varepsilon < x < a + \varepsilon$$



Theorem 23
Let $a \in \mathbb{R}$. If x belongs the neighborhood $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.

Proof
It holds $0 \leq |x - a| < \varepsilon$ for every $\varepsilon > 0$. Therefore by Theorem 11 $x - a = 0$. □

Example 24
If $x \in V_\varepsilon(a)$ and $y \in V_\varepsilon(b)$. Then by the triangle inequality

$$|(x + y) - (a + b)| = |(x - a) + (y - b)| \leq |x - a| + |y - b| \leq 2\varepsilon$$

Therefore $x + y \in V_{2\varepsilon}(a + b)$, i.e. $x + y$ is in the 2ε-neighborhood of $a + b$.

Lecture 6 (January 16)

2.3 Completeness of $\mathbb{R}$

In the introduction we have seen that

$$\sqrt{2} \notin \mathbb{Q},$$

which implies that $\mathbb{Q}$ is not *complete*. In fact $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ and as we will see $\mathbb{R}$ is *complete*, which one can think of that there are no *holes* in $\mathbb{R}$.
($\mathbb{R}$ is a complete (no holes) ordered (via smaller and bigger) field (one can calculate according to the field axioms), which makes it behave nicely.)

Definition 25
Let $\varnothing \neq S \subset \mathbb{R}$.

1. S is bounded above if there exists a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$. Each such u is called an upper bound for S.
2. S is bounded below if there exists a number $l \in \mathbb{R}$ such that $l \leq s$ for all $s \in S$. Each such l is called an lower bound for S.
3. A set is called bounded if it is bounded above and bounded below.
4. A set is called unbounded if it is not bounded.

Example 26
$S = \{x \in \mathbb{R} \,|\, x < 2\}$ is bounded above. 2 is an upper bound for S. 100 is an upper bound for S. It is not bounded below. So it is unbounded.

Every bounded set in $\mathbb{R}$ has infinitely many upper and lower bounds. The example shows that there is special one, the supremum.

Definition 27 (Supremum/Infimum)
Let $\varnothing \neq S \subset \mathbb{R}$.

1. If S is bounded above, then a number $u \in \mathbb{R}$ is called supremum or least upper bound of S if it satisfies
   (a) u is an upper bound for S
   (b) If v is any upper bound for S, then $u \leq v$
2. If S is bounded below, then a number $l \in \mathbb{R}$ is called infimum or greatest lower bound of S if it satisfies
   (a) l is an lower bound for S
   (b) If t is any lower bound for S, then $t \leq l$

Supremum of Infimum are unique, if they exist. We will write $u = \sup S$ if u is a supremum of S.
Note that $\varnothing \neq S \subset \mathbb{R}$ can have

- a supremum and infimum (e.g. $S = \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}$)

- a supremum and but no infimum (e.g. $S = \{x \in \mathbb{R} \,|\, x < 0\}$)
- an infimum but no supremum (e.g. $S = \{x \in \mathbb{R} \,|\, x > 1\}$)
- neither a supremum nor an infimum (e.g. $S = \mathbb{R}$ or $S = \mathbb{Z}$)

Lemma 28
Let $\varnothing \neq S \subset \mathbb{R}$ and u be an upper bound for S. u is the supremum of S if and only if for every $\varepsilon > 0$ there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Proof
" $\Longrightarrow$ ": Let $u = \sup S$ and $\varepsilon > 0$. Any $v = u - \varepsilon < u$ can not be an upper bound of S. Since otherwise v would be the supremum instead of u. Therefore there exists $s_\varepsilon \in S$ with $s_\varepsilon > v = u - \varepsilon$.
" $\Longleftarrow$ " Assume for every $\varepsilon > 0$ there is an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$. Since u is an upper bound we only have to show that there is no smaller upper bound. We will do this by contradiction: Assume there exists $v < u$ with $v \geq s$ for all $s \in S$. Then setting $\varepsilon = \frac{u-v}{2}$ we find $v = \frac{v+v}{2} \leq \frac{u+v}{2} = u - \frac{u-v}{2} = u - \varepsilon < s_\varepsilon \in S$, a contradiction to $v \geq s$ for all $s \in S$. draw picture! $\square$

Note that the supremum does not have to belong to the set. Both $S_1 = \{x \in \mathbb{R} \,|\, x < 1\}$ and $S_2 = \{x \in \mathbb{R} \,|\, x \leq 1\}$ have the same supremum, $\sup(S_1) = 1 = \sup(S_2)$.

Completeness Property of $\mathbb{R}$
Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

Remarks 29

- The important part is that the supremum is in $\mathbb{R}$.
- Unfortunately, we can not prove this property from the field axioms and Order properties of $\mathbb{R}$.

Definition 30 (Maximum/Minimum)
Let $\varnothing \neq S \subset \mathbb{R}$.

- $m \in S$ such that $m \geq s$ for all $s \in S$ is called maximum of S and we write $m = \max(S)$
- $m \in S$ such that $m \leq s$ for all $s \in S$ is called minimum of S and we write $m = \min(S)$

Difference between max **and** sup
Even though we did not introduce the maximum we have an intuition for it. One could define it for $\varnothing \neq S \subset \mathbb{R}$ as its biggest element. The difference between max and sup is that the maximum has to be an element of S, while the supremum does not have to be an element of S.

- $S = \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}$ $\quad \max S = 1$ $\quad \sup S = 1$
- $S = \{x \in \mathbb{R} \,|\, 0 \leq x < 1\}$ $\quad \max S$ does not exists $\quad \sup S = 1$
- $S = \left\{-\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\}$ $\quad \max S$ does not exists $\quad \sup S = 0$

2.4 Applications of the Supremum Property

Lemma 32
Let $\varnothing \neq S \subset \mathbb{R}$ be bounded above and $a \in \mathbb{R}$. Also define $a+S = \{a+s \mid s \in S\}$. Then

$$\sup(a+S) = a + \sup(S).$$

Proof
Let $u = \sup S$. Then $s \leq u$ for all $s \in S$ and therefore $a+s \leq a+u$ for all $s \in S$. This shows that $a+u$ is an upper bound of $a+S$, which implies that $\sup(a+S) \leq a+u$.
Let v be any upper bound of $a+S$. Then $a+s \leq v$ for all $s \in S$, implying $s \leq v-a$ for all $s \in S$. So $v-a$ is an upper bound of S. Therefore $u = \sup S \leq v-a$, implying $a+u \leq v$. But v was any upper bound so this holds for all upper bounds of $a+S$ and in particular $\sup(a+S)$. So $a+u \leq \sup(a+S)$.
Combining both estimates

$$a+u \leq \sup(a+S) \leq a+u,$$

implying $\sup(a+S) = a+u = a+\sup(S)$. □

In the second quiz you will handle multiplication.

Lecture 7 (January 20)

Lemma 33
Suppose $\varnothing \neq A, B \subset \mathbb{R}$ satisfy

$$a \leq b \text{ for all } a \in A \text{ and } b \in B.$$

Then

$$\sup A \leq \inf B.$$

We do not need that A is bounded from above and B from below, since this is automatically implied by the assumptions that both are nonempty and the inequality.

Proof
For any $\tilde{b} \in B$ we have that $a \leq \tilde{b}$ for all $a \in A$. Here I use $\tilde{b}$ to emphasize that we fix a specific, but arbitrary b. We could have omitted the $\tilde{}$. Therefore $\tilde{b}$ is an upper bound of A, i.e. $\sup A \leq \tilde{b}$.
So $\sup A \leq \tilde{b}$ for any $\tilde{b} \in B$. This means that $\sup A$ is a lower bound for B. As the infimum is the greatest lower bound in particular

$$\sup A \leq \inf B.$$

□

Functions

Definition 34
Given $f : D \to \mathbb{R}$, we say f is bounded (above/below) if the image of $f = f(D) = \{f(x) \,|\, x \in D\}$ is bounded (above/below).

Lemma 35
Suppose $f, g : D \to \mathbb{R}$ are bounded.

1. If $f(x) \le g(x)$ for all $x \in D$, then $\sup f(D) \le \sup g(D)$, which is sometimes written as
$$\sup_{x \in D} f(x) \le \sup_{x \in D} g(x)$$
2. $f(x) \le g(x)$ for all $x \in D$ does not necessarily imply $\sup f \le \inf g$.
3. $f(x) \le g(y)$ for all $x, y \in D$ does imply $\sup f \le \inf g$.

Proof

1. Since $f(x) \le g(x) \le \sup g$ for all $x \in D$ $\sup g$ is an upper bound for $f(D)$, and since the supremum is the smallest such upper bound $\sup f \le \sup g$.
2. Let $D = \{x \in \mathbb{R} \,|\, 0 \le x \le 2\}$ and $f(x) = x$, $g(x) = x + 1$. Then $\sup f = 2 > 1 = \inf g$.
3. Lemma 33 with $A = f(D)$ and $B = g(D)$ yields the claim.

□

Archimedean Property

As it turns out the Field Axioms and Order Properties of $\mathbb{R}$are not sufficient to prove that $\mathbb{N}$ is unbounded in $\mathbb{R}$. We need the completeness of $\mathbb{R}$.

Theorem 36 (Archimedean Property)
For any $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that $n \ge x$.

Proof
By contradiction, i.e. assume there exists $x \in \mathbb{R}$ such that $x > n$ for all $n \in \mathbb{N}$. Then x is an upper bound for $\mathbb{N}$. By the completeness property there exists $u = \sup \mathbb{N} \in \mathbb{R}$ and $u - 1 < \sup \mathbb{N}$. Thus $u - 1$ is not an upper bound, implying that there exists $m \in \mathbb{N}$ such that $u - 1 < m$. Adding 1 yields $u < m + 1 \in \mathbb{N}$, which is a contradiction that u is the supremum of N. □

Corollary 37
If $S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$, then $\inf S = 0$.

Proof
Since $S \neq \varnothing$ and S is bounded below by 0, it has an infimum w, which satisfies $w \geq 0$. For any $\varepsilon > 0$ the Archimedean Property implies that there exists an $n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < n$, which implies $\frac{1}{n} < \varepsilon$. Therefore

$$0 \leq w \leq \frac{1}{n} < \varepsilon$$

for any $\varepsilon > 0$, which by Theorem 11 implies $w = 0$. □

Corollary 38
For every $t > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < t$.

Proof
Since $\inf \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} = 0$ and $t > 0$, t can not be a lower bound for $\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. Thus there exists $\tilde{n} \in \mathbb{N}$ such that $0 < \frac{1}{\tilde{n}} < t$. □

Lecture 8 (January 22)

Roots
In the introduction have already seen that $2 \notin \mathbb{Q}$, but we never showed that $\sqrt{2}$ actually exists, which we will do now.

Theorem 39
There exists a unique $0 < x \in \mathbb{R}$ such that $x^2 = 2$.

Proof
We want to use some existence result. And the only suitable one that we have is the Completeness Property of $\mathbb{R}$, which says that every nonempty set of real numbers that has an upper bound has a supremum in $\mathbb{R}$. So the proof is to construct a set that fulfills the assumptions. Let $S = \left\{ s \in \mathbb{R} \,\middle|\, 0 \leq s, s^2 < 2 \right\}$. Since $1 \in S$, the set is not empty. It is also bounded above by 2 since if $t > 2$, then $t^2 > 2t > 4$, showing that $t \notin S$. So we can apply the Completeness Property and get that S has a supremum in $\mathbb{R}$, which we call $x (= \sup S)$.
We have to show that $x^2 = 2$, which we will do by proving $x^2 \not< 2$ and $x^2 \not> 2$ via contradiction. Showing that something does not fulfill a certain condition usually hints at a proof by contradiction.
The second condition of S, i.e. $s^2 < 2$ indicates that the supremum satisfies $x^2 = 2$, so the idea is that if it is smaller than this we could squeeze another $s = x + \frac{1}{n} \in S$ in between. Similar if the supremum was bigger then we could find a lower upper bound $x - \frac{1}{m}$. Now we only have to write this mathematically.
Assume $x^2 < 2$. Then for any $n \in \mathbb{N}$

$$\left(x + \frac{1}{n} \right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{2x}{n} + \frac{1}{n} = x^2 + \frac{2x+1}{n}. \tag{9}$$

We want right hand side to be < 2, since then $x + \frac{1}{n} \in S$. So we calculate

$$x^2 + \frac{2x+1}{n} \overset{!}{<} 2 \qquad \Leftrightarrow \qquad \frac{2x+1}{2-x^2} < n$$

The Archimedean Property says there exists $\tilde{n} > \frac{2x+1}{2-x^2}$. Therefore

$$\left(x + \frac{1}{\tilde{n}}\right)^2 \overset{(9)}{\leq} x^2 + \frac{2x+1}{\tilde{n}} < x^2 + 2 - x^2 = 2,$$

showing that $x < x + \frac{1}{\tilde{n}} \in S$, so x is not an upper bound for S, so $x \neq \sup S$, a contradiction.
Assume $x^2 > 2$. Then for any $m \in \mathbb{N}$

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}. \tag{10}$$

Again we want the right hand side to be > 2 since then $x - \frac{1}{m}$ is a better supremum.

$$x^2 - \frac{2x}{m} \overset{!}{>} 2 \qquad \Leftrightarrow \qquad x^2 - 2 > \frac{2x}{m} \qquad \Leftrightarrow \qquad m > \frac{2x}{x^2-2}.$$

The Archimedean Property says there exists $\tilde{m} > \frac{2x}{x^2-2}$. Therefore

$$\left(x - \frac{1}{\tilde{m}}\right)^2 \overset{(10)}{\geq} x^2 - \frac{2x}{\tilde{m}} > 2 > s^2 \tag{11}$$

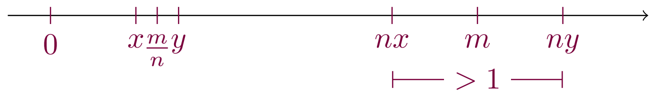
for any $s \in S$. By Theorem 15 (For $0 \leq a, b \in \mathbb{R}$ one has $a < b \Leftrightarrow a^2 < b^2$), (11) shows that $x - \frac{1}{\tilde{m}} > s$ for every $s \in S$, i.e. $x - \frac{1}{\tilde{m}}$ is an upper bound for S, a contradiction to x being the supremum (lowest upper bound).
Uniqueness: Uniqueness is usually shown in this way: Suppose there are 2 that work, then they have to coincide. Suppose $0 < x, y \in \mathbb{R}$ satisfy $x^2 = 2$, $y^2 = 2$. Then $(x+y)(x-y) = x^2 - y^2 = 2 - 2 = 0$. Dividing by $x + y$ shows $x - y = 0$, i.e. $x = y$. □

Theorem 39 shows that x is well defined. Therefore $\sqrt{2} := x$. Similarly one can show that the square root of any positive number r is uniquely determined in $\mathbb{R}$, i.e. $x = \sqrt{r}$ is defined as the unique solution to $0 < x \in \mathbb{R}$ satisfying $x^2 = r$.

Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 40 ($\mathbb{Q}$ is dense in $\mathbb{R}$.)
Let $x, y \in \mathbb{R}$ satisfy $x < y$. Then there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Proof
The idea is that if we multiply x, y by a big natural number such that there is a natural number in between. Then dividing by that number yields the result.



Without loss of generality assume $x, y > 0$. (Otherwise first prove for positive and then $x < r < y \Leftrightarrow x+a < r+a < y+a$ for large $a \in \mathbb{Q}$ such that $x+a > 0$.) Calculating

$$ny - nx > 1 \qquad \Leftrightarrow \qquad y - x > \frac{1}{n}$$

By the Archimedean Property there exists $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < y - x$$

and therefore

$$1 < ny - nx. \tag{12}$$

Now we want to find m in between ny and nx Let $k \in \mathbb{N}$ be the smallest natural number such that

$$ny \leq k, \tag{13}$$

which exists due to the Archimedean Property. This also implies

$$k - 1 < ny, \tag{14}$$

since otherwise $\tilde{k} = k - 1$ would be the smallest number in $\mathbb{N}$ such that $ny \leq \tilde{k}$. Here it is helpful that $ny > 0$ by the without loss of generality statement since otherwise there is a gap between $ny \leq 0 < k$ Combining everything

$$nx \overset{(12)}{<} ny - 1 \overset{(13)}{\leq} k - 1 \overset{(14)}{<} ny.$$

Therefore $m = k - 1 \in \mathbb{N}_0$ satisfies

$$nx < m < ny$$

and dividing by n and defining $r = \frac{m}{n}$ we find

$$x < r < y,$$

where $r = \frac{m}{n} \in \mathbb{Q}$. □

Lecture 9 (January 23)

Corollary 41
If $x, y \in \mathbb{R}$ and $x < y$, then there exists an irrational number z such that $x < z < y$.

Proof
We know that there is a rational number r in between them so if we can smuggle in a factor of $\sqrt{2}$, we find a irrational number. So we want to find

$$x < \sqrt{2}r < y \qquad \Leftrightarrow \qquad \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}},$$

where r is rational. Similar to before without loss of generality $x, y > 0$. By Theorem 40 there exists $0 < r \in Q$ such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Multiplying by $\sqrt{2}$ we find

$$x < \sqrt{2}r < y.$$

We found a candidate $z = \sqrt{2}r$. It is left to show that $\sqrt{2}r \notin \mathbb{Q}$. By contradiction if $\sqrt{2}r \in \mathbb{Q}$ then $\sqrt{2}r = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$. But since r is rational $0 \neq r = \frac{k}{l}$, implying

$$\sqrt{2} = \frac{m}{n}\frac{1}{r} = \frac{ml}{nk} \in \mathbb{Q},$$

a contradiction to Theorem 1 ($\sqrt{2} \notin \mathbb{Q}$). □

Theorem 40 and Corollary 41 show that we can find rational and irrational numbers in between every 2 (different) numbers. Applying the Theorem/Corollary repeatedly there exist infinite numbers of both kinds between every different numbers. As we will see later there are "more" irrational (uncountable many) than rational (countable many) numbers.

2.5 Intervals

Definition 42
Let $a, b \in \mathbb{R}$ with $a < b$. Then

- $(a, b) := \{x \in \mathbb{R} \,|\, a < x < b\}$ is called an <u>open interval</u>
- $[a, b] := \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$ is called an <u>closed interval</u>
- $[a, b) := \{x \in \mathbb{R} \,|\, a \leq x < b\}$ and $(a, b] := \{x \in \mathbb{R} \,|\, a < x \leq b\}$ are called <u>half-open</u> or <u>half-closed intervals</u>

- $$\begin{aligned}(a,\infty) &:= \{x \in \mathbb{R} \,|\, x > a\} \\ [a,\infty) &:= \{x \in \mathbb{R} \,|\, x \geq a\} \\ (-\infty,b) &:= \{x \in \mathbb{R} \,|\, x < b\} \\ (-\infty,b] &:= \{x \in \mathbb{R} \,|\, x \leq b\} \\ (-\infty,\infty) &:= \mathbb{R}\end{aligned}$$

Remarks 43

- $(a,a) = [a,a) = (a,a] = \varnothing$ and $[a,a] = \{a\}$ are not intervals.
- ∞ is not a number, (a,∞) is just notation for $\{x \in \mathbb{R} \,|\, x > a\}$. In general ∞ breaks things, be careful when and how to use it!

When is a set an interval?

Theorem 44
Assume $S \subset \mathbb{R}$ has at least 2 points and

$$\text{for all } x,y \in S, x < y \implies [x,y] \subset S \tag{15}$$

holds. Then S is an interval.

This means if any point in between two points of the set is in the set then the set is an interval.

Proof
We need to show that either of the interval definitions applies, i.e. either $S = [a,b]$, $S = (a,b)$, ... We will show that $(a,b) \subset S \subset (a,b)$ or the (half-) closed versions of this. We look at different cases

- Assume S is bounded. Then there exists $a = \inf S$ and $b = \sup S$ and $S \subset [a,b]$. Next we will show that $(a,b) \subset S$, i.e. every z with $a < z < b$ fulfills $z \in S$. z can not be a lower bound (since otherwise $\inf S = z$), nor can it be an upper bound (since otherwise $\sup S = z$). Therefore there exists $x,y \in S$ with $x < z < y$, i.e. $z \in (x,y)$ for some $x,y \in S$. By (15) we get $z \in (x,y) \subset [x,y] \subset S$. Showing that $z \in S$, which was arbitrary, so it holds for every $z \in (a,b)$, i.e. $(a,b) \subset S$.
  - If $a,b \in S$ then $S \subset [a,b] = \{a,b\} \cup (a,b) \subset S$
  - If $a \in S$, $b \notin S$ then $S \subset [a,b] \setminus \{b\} = \{a\} \cup (a,b) \subset S$
  - If $b \in S$, $a \notin S$ then $S \subset [a,b] \setminus \{a\} = \{b\} \cup (a,b) \subset S$
  - If $a,b \notin S$ then $S \subset [a,b] \setminus \{a,b\} = (a,b) \subset S$
- Assume S is above but not below. Let $b = \sup S$. Then $S \subset (-\infty,b]$ and we will show that $(-\infty,b) \subset S$, i.e. every $z \in \mathbb{R}$ with $z < b$ fulfills $z \in S$. z can not be a lower bound since S is not bounded below and z is not an upper bound (since otherwise $\sup S = z$). Therefore there exists $x,y \in S$ with $x < z < y$, i.e. $z \in (x,y)$ for some $x,y \in S$. By (15) we get $z \in (x,y) \subset [x,y] \subset S$. Showing that $z \in S$, which was arbitrary, so it holds for every $z \in (-\infty,b)$, i.e. $(-\infty,b) \subset Z$.

- If $b \in S$ then $S \subset (-\infty, b) \subset (-\infty, b) \cup \{b\} \subset S$
- If $b \notin S$ then $S \subset (-\infty, b] \setminus \{b\} = (-\infty, b) \subset S$

- say similar to before Assume S is below but not above. Let $a = \inf S$. Then $S \subset [a, \infty)$ and we will show that $(a, \infty) \subset S$, i.e. every $z \in \mathbb{R}$ with $a < z$ fulfills $z \in S$. z can not be an upper bound since S is not bounded above and z is not a lower bound (since otherwise $\inf S = z$). Therefore there exists $x, y \in S$ with $x < z < y$, i.e. $z \in (x, y)$ for some $x, y \in S$. By (15) we get $z \in (x, y) \subset [x, y] \subset S$. Showing that $z \in S$, which was arbitrary, so it holds for every $z \in (-\infty, b)$, i.e. $(-\infty, b) \subset Z$.
  - If $a \in S$ then $S \subset (a, \infty) \subset (a, \infty) \cup \{a\} \subset S$
  - If $a \notin S$ then $S \subset (-\infty, a] \setminus \{a\} = (a, \infty) \subset S$

- say similar to before Assume S is neither bounded below nor bounded above. We will show that $(-\infty, \infty) \subset S$, i.e. every $z \in \mathbb{R}$ fulfills $z \in S$. z can neither be an upper nor a lower bound since S is neither bounded above nor below. Therefore there exists $x, y \in S$ with $x < z < y$, i.e. $z \in (x, y)$ for some $x, y \in S$. By (15) we get $z \in (x, y) \subset [x, y] \subset S$. Showing that $z \in S$, which was arbitrary, so it holds for every $z \in (-\infty, \infty)$, i.e. $(-\infty, \infty) \subset Z \subset \mathbb{R} = (-\infty, \infty)$.

□

Lecture 10 (January 27)

Nested Intervals

Definition 45 (Nested Intervals)
We say that a sequence of intervals I_n, $n \in \mathbb{N}$ is nested if for every $n \in \mathbb{N}$ one has $I_{n+1} \subset I_n$, i.e. the following chain of inclusions holds

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \ldots.$$

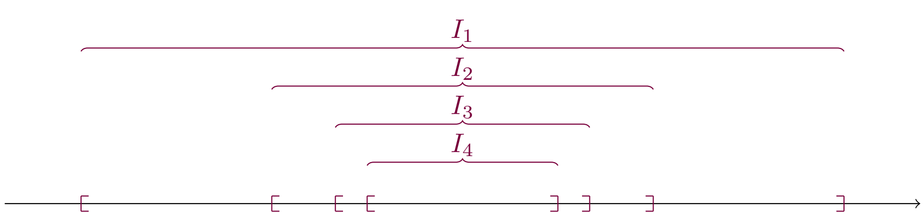


Figure 1: Nested Intervals

Example 46
We define $x \in \bigcap_{n=1}^{\infty} I_n \Leftrightarrow x \in I_n$ for all $n \in \mathbb{N}$. There is no number ∞ here. It is just notation.

- For $I_n = \left[0, \frac{1}{n}\right]$ one has $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, so these are nested intervals. And $\bigcap_{n=1}^{\infty} I_n = \{0\}$. (Proof in assignment)
- $I_n = \left(0, \frac{1}{n}\right)$ again are nested intervals but $\bigcap_{n=1}^{\infty} I_n = \varnothing$. (Proof in assignment)

Theorem 47 (Nested Intervals Property)
If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, then there exists a number $x \in \mathbb{R}$ such that $x \in I_n$ for all $n \in \mathbb{N}$, i.e. $x \in \bigcap_{n=1}^{\infty} I_n$.

Proof
We will show that $\sup\{a_k \in \mathbb{R} \,|\, k \in \mathbb{N}\}$ does the job. On one side it is $\geq$ than the left end points, on the other side it is $\leq$ then the right end points. For any $k \in \mathbb{N}$ we have $a_k < b_k \leq b_1$. Thus b_1 is an upper bound for the set $\{a_k \in \mathbb{R} \,|\, k \in \mathbb{N}\}$, which is non-empty. By the Completeness Property of $\mathbb{R}$ there exists a supremum of this set, which we call x. This implies that $a_n \leq \sup\{a_k \in \mathbb{R} \,|\, k \in \mathbb{N}\} = x$.
Next we'll show that $x \leq b_n$ for all $n \in \mathbb{N}$. We will first show that for every $n \in \mathbb{N}$, b_n is an upper bound for the set $\{a_k \in \mathbb{R} \,|\, k \in \mathbb{N}\}$.

- If $n \leq k$, then since $I_n \supset I_k$ we have $a_k \leq b_k \leq b_n$.
- If $k < n$, then since $I_n \subset I_k$ we have $a_k \leq a_n \leq b_n$.

Thus $a_k \leq b_n$ for all $k \in \mathbb{N}$, i.e. b_n is an upper bound for $\{a_k \in \mathbb{R} \,|\, k \in \mathbb{N}\}$. Therefore $a_n \leq x = \sup\{a_k \in \mathbb{R} \,|\, k \in \mathbb{N}\} \leq b_n$, implying $x \in I_n$. $\square$

One can show that if the width of the intervals shrinks, i.e. $\inf\{b_n - a_n \,|\, n \in \mathbb{N}\} = 0$, then there is only one such x, i.e. it is unique.

3 Sequences and Series

3.1 Sequences and Their Limits

Definition 48 (Sequence)
A sequence of real numbers is a function $X : \mathbb{N} \to \mathbb{R}$. We will write

$$x_n = X(n)$$

for individual values and denote the whole sequence by

$$X, \qquad (x_n), \qquad (x_n)_{n\in\mathbb{N}} \quad \text{or} \quad (x_n \mid n \in \mathbb{N}).$$

Remark 49 (Recursively defined sequences)
A sequence can also be defined recursively (or inductively) by specifying $x_1, \dots, x_k$ for $k \in \mathbb{N}$ and $x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k})$ where $f : \mathbb{R}^k \to \mathbb{R}$.

Example 50

- $(a)_{n\in\mathbb{N}} = (a, a, a, \dots)$ for $a \in \mathbb{R}$
- $(2)_{n\in\mathbb{N}} = (2, 2, 2, \dots)$
- $(n^2)_{n\in\mathbb{N}} = (1^2, 2^2, 3^2, \dots) = (1, 4, 9, \dots)$
- $(n^k)_{k\in\mathbb{N}} = (n^1, n^2, n^3, \dots)$ for $n \in \mathbb{R}$
- $(2n)_{n\in\mathbb{N}} = (2, 4, 6, \dots)$, which is the same as the recursively defined

$$x_1 = 2, \qquad x_{n+1} = x_n + 2$$

- The Fibonacci sequence

$$f_1 = f_2 = 1, \qquad f_{n+1} = f_n + f_{n-1}$$

Limits

Definition 51 (Convergence/Limit)
A sequence (x_n) of real numbers converges to $x \in \mathbb{R}$, equivalently x is the limit of the sequence (x_n) if for every $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$ one has

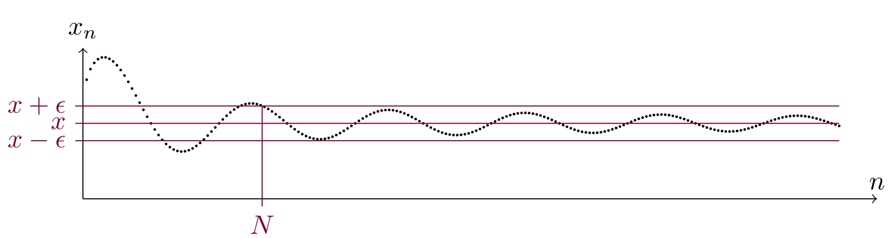
$$|x_n - x| < \varepsilon.$$

If the sequence converges we write

$$\lim_{n\to\infty}(x_n) = x, \qquad x_n \xrightarrow{n\to\infty} x \qquad \text{or} \qquad x_n \to x,\ \text{as } n \to \infty.$$

If the sequence does not converge, i.e. there exists no such limit x, it is said to diverge.

Lecture 11 (January 29)



Remarks 52

- N may depend on ε.
- The condition $|x_n - x| < \varepsilon$ could be changed to $|x_n - x| \leq \varepsilon$.

$$|x_n - x| < \varepsilon \implies |x_n - x| \leq \varepsilon$$
$$|x_n - x| \leq \varepsilon < 2\varepsilon =: \tilde{\varepsilon} \quad \forall \tilde{\varepsilon} > 0 \Longleftarrow |x_n - x| \leq \varepsilon \quad \forall \varepsilon > 0$$

- Usually ε being small is the difficult case, in which most likely we require N to be big.
- Again $n \to \infty$ is just notation for the definition, which does not involve any ∞. We do not plug in $n = \infty$.

The Collatz-Conjecture is a easy to understand open problem for a sequence.

Theorem 53 (Uniqueness of Limits)
A sequence in $\mathbb{R}$ can have at most one limit.

Proof
Suppose $x, \tilde{x}$ are both limits of (x_n). For each $\varepsilon > 0$ there exists $N, \tilde{N} \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon, \qquad |x_{\tilde{n}} - \tilde{x}| < \varepsilon$$

for $n \geq N$, $\tilde{n} \geq \tilde{N}$. Therefore for $n \geq \max\{N, \tilde{N}\}$ by the triangle inequality

$$|x - \tilde{x}| = |x - x_n + x_n - \tilde{x}| \leq |x - x_n| + |x_n - \tilde{x}| < 2\varepsilon.$$

Defining $\tilde{\varepsilon} = \frac{\varepsilon}{2}$ we find for any $\tilde{\varepsilon} > 0$ that

$$0 \leq |x - \tilde{x}| < \tilde{\varepsilon},$$

which by Theorem 11 implies that $|x - \tilde{x}| = 0$. Thus $x = \tilde{x}$. □

Lemma 54
Let (x_n) be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. Then the following are equivalent

1. (x_n) converges to x
2. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

   $$|x_n - x| < \varepsilon$$

   holds.
3. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

   $$x - \varepsilon < x_n < x + \varepsilon$$

   holds.

4. For every ε-neighborhood $V_\varepsilon(x)$ of x there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$x_n \in V_\varepsilon(x)$$

holds.

Proof
1 is equivalent to 2 by the Definition of the limit. The equivalence of 2 and 3 follows analogous to Theorem 18.3 (which handled $\leq$ instead of $<$). 3 is equivalent to 4 by Definition 22 (ε-neighborhoods). □

Remarks 55

- These statements could also be written in reverse: *For every $\varepsilon > 0$ only a finite number of elements x_n are not in $V_\varepsilon(x)$.* These finite number of elements that are (potentially) not in $V_\varepsilon(x)$ are $x_1, \ldots, x_{N-1}$.
- The definition of a limit is not constructive. In practice we try to guess the limit and then prove that it is the limit.

Example 56

1. $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

   Proof
   We want

   $$\left|\frac{1}{n} - 0\right| < \varepsilon \quad \Leftrightarrow \quad \frac{1}{\varepsilon} < n$$

   and if we increase n the left side shrinks more. Given $\varepsilon > 0$ the Archimedean Property asserts that there exists $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$. Therefore for any $n \geq N$

   $$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \overset{n\geq N}{\leq} \frac{1}{N} \overset{\frac{1}{\varepsilon}<N}{<} \varepsilon.$$

   Thus $\frac{1}{n}$ converges to 0. □

2. $\lim_{n\to\infty}\left(\frac{3n^2}{n^2-5n+1}\right) = 3$

   Proof
   We want to estimate so that the term still goes to 0, i.e.

   $$\left|\frac{3n^2}{n^2-5n+1} - 3\right| = \left|\frac{3n^2-3n^2+15n-3}{n^2-5n+1}\right| = \left|\frac{15n-3}{n^2-5n+1}\right|$$
   $$\overset{\text{if } n\geq 5}{=} \frac{15n-3}{n^2-5n+1} < \frac{15n}{n^2-5n} = \frac{15}{n-5} \overset{!}{<} \varepsilon \tag{16}$$
   $$\Leftrightarrow \quad \frac{15}{\varepsilon} < n-5 \quad \Leftrightarrow \quad \frac{15}{\varepsilon} + 5 < n$$

Given $\varepsilon > 0$ the Archimedean Property asserts that there exists $N \in \mathbb{N}$ such that $N > 5 + \frac{15}{\varepsilon}$. Therefore for any $n \geq N$

$$\left|\frac{3n^2}{n^2 - 5n + 1} - 3\right| \overset{(16)}{<} \frac{15}{n-5} \overset{n\geq N}{\leq} \frac{15}{N-5} \overset{N>5+\frac{15}{\varepsilon}}{<} \varepsilon$$

Thus $\frac{3n^2}{n^2-5n+1}$ converges to 3. □

Lecture 12 (January 30)

3. Find the limit of $x_n = \sqrt{n+1} - \sqrt{n}$.

Solution
$\sqrt{101} - \sqrt{100} = 0.049\ldots$ or use the graph of $\sqrt{x}$ to see that the values get closer and closer. Guess $\lim x_n = 0$.

$$\begin{aligned}\left|\sqrt{n+1} - \sqrt{n} - 0\right| &= \left|\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}\right| = \left|\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}\right| \\ &= \left|\frac{1}{\sqrt{n+1} + \sqrt{n}}\right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}} \overset{!}{<} \varepsilon \qquad (17) \\ \Leftrightarrow \frac{1}{\varepsilon^2} < n\end{aligned}$$

For $\varepsilon > 0$, let $N \in \mathbb{N}$ such that $N > \varepsilon^2$. For $n \geq N$ by (17)

$$\left|\sqrt{n+1} - \sqrt{n} - 0\right| \leq \frac{1}{\sqrt{n}} \overset{n\geq N}{\leq} \frac{1}{\sqrt{N}} < \varepsilon,$$

proving $\lim \sqrt{n+1} - \sqrt{n} = 0$. □

4. Prove that the sequence (x_n) with $x_n = (-1)^n$ diverges.

Proof
By contradiction. Assume there exits $x \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$

$$|(-1)^n - x| = |x_n - x| < \varepsilon.$$

Then setting $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that

$$|(-1)^n - x| < 1$$

holds for all $n \geq N$. If n is odd then

$$|-1-x| = |(-1)^n - x| < 1 \quad \Leftrightarrow \quad -1 < -1-x < 1$$
$$\Leftrightarrow \quad -2 < x < 0$$

And if n is even

$$|1-x| = |(-1)^n - x| < 1 \quad \Leftrightarrow \quad -1 < 1-x < 1$$
$$\Leftrightarrow \quad 0 < x < 2.$$

So $x < 0$ and $x > 0$, a contradiction. □

Lemma 57
Let $m \in \mathbb{N}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence. $(x_n)_{n\in\mathbb{N}}$ converges if and only if $(\tilde{x}_n)_{n\in\mathbb{N}} = (x_{m+n})_{n\in\mathbb{N}}$ converges. If they converge $\lim_{n\to\infty}(x_n) = \lim_{n\to\infty}(x_{m+n})$.

$(x_{m+n})_{n\in\mathbb{N}}$ is called the m-tail of $(x_n)_{n\in\mathbb{N}}$.

Proof
" $\Longrightarrow$ "
Assume that (x_n) converges to x. Let $\varepsilon > 0$. If we find an $N \in \mathbb{N}$ such that $|\tilde{x}_n - x| < \varepsilon$ for all $n \geq N$ we are done. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ one has

$$|x_n - x| < \varepsilon.$$

In particular this holds for all $k := m + n > n \geq N$, implying

$$|\tilde{x}_n - x| = |x_{m+n} - x| = |x_k - x| < \varepsilon$$

for any $n \geq N$, i.e. $\tilde{x}_n$ converges to x.
" $\Longleftarrow$ "
Assume that $(\tilde{x}_n)$ converges to x. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $k \geq N$ one has

$$|\tilde{x}_k - x| < \varepsilon.$$

Setting $n = m + k$ we find that

$$|x_n - x| = |x_{m+k} - x| = |\tilde{x}_k - x| < \varepsilon$$

for all $n = m + k > m + N = \tilde{N}$, i.e. (x_n) converges to x. □

Definition 58 (Almost all/Ultimately)
We say that a sequence (x_n) has a property for almost all elements if it has the property for all but finitely many elements. In that case we also say the sequence has ultimately this property.

Example 59

- $\frac{1}{n} < \frac{1}{100}$ for almost all elements (or almost every $n \in \mathbb{N}$).
- $(3, 3, 4, 3, 2, 3, 3, 3, 3, 3, 3, 3, \dots)$ is ultimately constant.

Next we formalize what we have done already when proving convergence.

Theorem 60 (Squeeze/Sandwich Theorem 1)
Let (x_n) be a sequence in $\mathbb{R}$, $x \in \mathbb{R}$ and (a_n) be a sequence of positive real numbers with $\lim_{n\to\infty}(a_n) = 0$. If there exists $0 < C \in \mathbb{R}$ and $N \in \mathbb{N}$ such that

$$|x_n - x| \leq C a_n$$

for all $n \geq N$, then $\lim_{n\to\infty}(x_n) = x$.

Proof
For any $\varepsilon > 0$ there exists $\tilde{N} \in \mathbb{N}$ such that

$$a_n = |a_n - 0| < \varepsilon$$

for all $n \geq \tilde{N}$. Therefore

$$|x_n - x| \leq C a_n < C\varepsilon$$

for $n \geq \max\{N, \tilde{N}\}$. Fix by starting with $\frac{\varepsilon}{C}$ instead of ε. For any $\frac{\varepsilon}{C} = \tilde{\varepsilon} > 0$ there exists $\tilde{N} \in \mathbb{N}$ such that

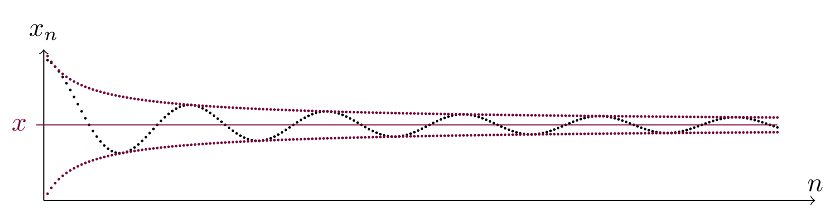
$$a_n = |a_n - 0| < \tilde{\varepsilon}$$

for all $n \geq \tilde{N}$. Therefore

$$|x_n - x| \leq C a_n < C\tilde{\varepsilon} = C\frac{\varepsilon}{C} = \varepsilon$$

for all $n \geq \max\{N, \tilde{N}\}$, implying $\lim_{n\to\infty}(x_n) = x$. □

Lecture 13 (February 03)



Example 61
Show that $\lim_{n\to\infty}\left(\frac{\sin n}{n}\right) = 0$ We did not properly define sin but it illustrates the technique.

Sketch of Proof
Since $-1 \le \sin(n) \le 1$ for all $n \in \mathbb{N}$ we find that

$$|x_n - 0| = \left|\frac{\sin n}{n}\right| = \frac{|\sin n|}{|n|} \le \frac{1}{n} \to 0.$$

□

3.2 Limit Theorems

Definition 62
A sequence (x_n) of real numbers is said to be bounded if there exists $0 < M \in \mathbb{R}$ such that $|x_n| \le M$ for all $n \in \mathbb{N}$.

Since the first finitely many elements are real numbers, these are bounded. So we can disregard these and *only* care about high values of n.

Theorem 63
A convergent sequence of real numbers is bounded.

Proof
Let $x = \lim(x_n)$ and $\varepsilon = 1$. Then there exists a natural number N such that $|x_n - x| < 1$ holds for all $n \ge N$. Therefore

$$|x_n| = |x_n - x + x| \overset{\text{triangle}}{\le} |x_n - x| + |x| \le 1 + |x|.$$

Defining

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x|\}$$

which exists as the maximum of a finite set it follows that

$$|x_n| \le M$$

for all $n \in \mathbb{N}$. □

End of material for Test 1.
From now on we will always assume that (x_n), (y_n), ... will be sequences of real numbers.

Theorem 64 (Linearity of Limits)
Let (x_n) and (y_n) be a sequence of real numbers and converge to x and y respectively and $c \in \mathbb{R}$. Then

1. $\lim(x_n + y_n) = x + y$

2. $\lim(x_n - y_n) = x - y$

3. $\lim(x_n y_n) = xy$

4. $\lim(cx_n) = cx$

5. If $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y \neq 0$, then $\lim\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$

Proof

1. For $\frac{\tilde{\varepsilon}}{2} = \varepsilon > 0$ there exists $N, \tilde{N} \in N$ such that for all $n \geq N$, $\tilde{n} \geq \tilde{N}$

$$|x_n - x| < \varepsilon = \frac{\tilde{\varepsilon}}{2}, \qquad |y_{\tilde{n}} - y| < \varepsilon = \frac{\tilde{\varepsilon}}{2}$$

Therefore for $n \geq \max\{N, \tilde{N}\}$

$$|x_n + y_n - (x + y)| \leq |x_n - x| + |y_n - y| < 2\varepsilon = \tilde{\varepsilon}$$

2. Setting $\tilde{y}_n = -y_n$ this follows directly by Part 1 by noticing that $\lim \tilde{y}_n = -y$ since $|\tilde{y}_n - (-y)| = |-(y_n - y)| = |y_n - y|$.

3. Let $\varepsilon > 0$ be arbitrary. Then there exist $N, \tilde{N} \in \mathbb{N}$ such that for all $n \geq \tilde{N}$, $\tilde{n} \geq \tilde{N}$

$$|x_n - x| < \varepsilon, \qquad |y_{\tilde{n}} - y| < \varepsilon.$$

For $n \geq \max\{N, \tilde{N}\}$

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| = |x_n(y_n - y) + (x_n - x)y| \\ &\leq |x_n(y_n - y)| + |(x_n - x)y| \leq |x_n|\,|y_n - y| + |x_n - x|\,|y| \end{aligned}$$

$|y_n - y|$ and $|x_n - x|$ behave nicely (the are small). $|y| \in \mathbb{R}$ is just a number, so we can treat it like the 2 in Part 1. The only problem is $|x_n|$ could grow as n increases, but this is impossible as it is bounded since it converges. By Theorem 63 there exists $0 < M \in \mathbb{R}$ such that $|x_n| \leq M$, implying

$$|x_n y_n - xy| \leq \underbrace{|x_n|}_{\leq M}\,\underbrace{|y_n - y|}_{<\varepsilon} + \underbrace{|x_n - x|}_{<\varepsilon}\,|y| < (M + |y|)\varepsilon.$$

Again starting with $\tilde{\varepsilon} = \frac{\varepsilon}{M+|y|}$ instead of ε we get

$$|x_n y_n - xy| < (M + |y|)\tilde{\varepsilon} = \varepsilon.$$

We can always fix these proofs as long as the prefactor (here $M + |y|$) is independent of n! So it is important to estimate until all n dependency is gone!

4. Follows by Part 3 with the constant sequence $(y_n)_{n\in\mathbb{N}} = (c)_{n\in\mathbb{N}}$

5. Follows by Part 3 with $(\tilde{y}_n) = \left(\frac{1}{y_n}\right)$ after showing that $\lim \left(\frac{1}{y_n}\right) = \frac{1}{y}$.

Starting to estimate we find

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y}{yy_n} - \frac{y_n}{y_n y}\right| = \frac{1}{|yy_n|}|y - y_n|$$

$|y - y_n| \to 0$ and $\frac{1}{|y|} = \text{const}$ are nice terms, and only $\frac{1}{y_n}$ could make problems. But it should be close to $\frac{1}{y}$ for big n. They can not be equal but the following could be true

$$\frac{1}{|y_n|} \overset{!}{\leq} \frac{2}{|y|}$$

$$\Leftrightarrow \qquad |y| \leq 2|y_n| \qquad \Leftrightarrow \qquad \frac{|y|}{2} \leq |y_n|$$

let's try to accomplish this

$$|y| = |y - y_n + y_n| \leq |y - y_n| + |y_n| < \varepsilon + |y_n| \overset{\varepsilon < \frac{|y|}{2}}{<} \frac{|y|}{2} + |y_n|$$

$$\Leftrightarrow \qquad \frac{|y|}{2} < |y_n|$$

so it works

So we can estimate both terms by paying the price of a factor of 2. Now we just have to put everything together.

Lecture 14 (February 05)

Let arbitrary $\varepsilon > 0$ be given. For any $\tilde{\varepsilon} > 0$, there exists $N(\tilde{\varepsilon}) \in \mathbb{N}$ such that for all $\tilde{n} \geq N(\tilde{\varepsilon})$

$$|y_{\tilde{n}} - y| < \tilde{\varepsilon}. \tag{18}$$

So for $\tilde{\varepsilon} = \frac{|y|}{2}$ there exists $N(\frac{|y|}{2}) \in \mathbb{N}$ such that for all $n \geq N(\frac{|y|}{2})$

$$|y_n - y| < \frac{|y|}{2},$$

implying

$$|y| = |y - y_n + y_n| \leq |y - y_n| + |y_n| < \frac{|y|}{2} + |y_n|.$$

Thus

$$\frac{|y|}{2} < |y_n| \quad \text{or} \quad \frac{1}{|y_n|} < \frac{2}{|y|} \tag{19}$$

for all $n \geq N(\frac{|y|}{2})$. Therefore for $n \geq \max\{N(\frac{|y|}{2}), N(\tilde{\varepsilon})\}$

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y y_n}\right| = \frac{1}{|y|}\frac{1}{|y_n|}|y_n - y| \overset{(19)}{<} \frac{2}{|y|^2}|y_n - y| \overset{(18)}{<} \frac{2}{|y|^2}\tilde{\varepsilon}. \quad (20)$$

This holds for any $\tilde{\varepsilon} > 0$. Choosing $\tilde{\varepsilon} = \frac{y^2}{2}\varepsilon$, we find that for any $n \geq \tilde{N} := \max\{N(\frac{|y|}{2}), N(\frac{y^2}{2}\varepsilon)\}$

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| \overset{(20)}{<} \frac{2}{|y|^2}\tilde{\varepsilon} = \frac{2}{|y|^2}\frac{|y|^2}{2}\varepsilon = \varepsilon.$$

Since for any arbitrary $\varepsilon > 0$ we can find $\tilde{N}$ such that for all $n \geq \tilde{N}$

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| < \varepsilon$$

holds we conclude that $\lim\left(\frac{1}{y_n}\right) = \frac{1}{y}$.

□

Theorem 65
If (x_n) is convergent and $x_n \geq 0$ for (almost) all $n \in \mathbb{N}$. Then $\lim(x_n) \geq 0$.

Proof
By contradiction, i.e. assume $x = \lim(x_n) < 0$. Since (x_n) converges to x, for every $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon$$

or equivalently

$$x - \varepsilon < x_n < x + \varepsilon$$

for all $n \geq N$. In particular for $\varepsilon = \frac{|x|}{2}$ it holds

$$x_n < x + \frac{|x|}{2} \leq \frac{x}{2} < 0$$

for all $n \geq N$, a contradiction to $x_n \geq 0$ for (almost) all $n \in \mathbb{N}$. □

Note that Theorem 65 does not hold when one replaces the $\geq$ by $>$, as can be seen by $\lim\left(\frac{1}{n}\right) = 0$ even though $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

Theorem 66
Let (x_n) and (y_n) be convergent and $x_n \leq y_n$ for (almost) all $n \in \mathbb{N}$. Then

$$\lim(x_n) \leq \lim(y_n).$$

Proof
Defining $(z_n) = (y_n - x_n)$ one has $z_n = y_n - x_n \geq 0$ and Theorem 65 implies

$$\lim(z_n) \geq 0. \tag{21}$$

By Theorem 64

$$\lim(y_n) - \lim(x_n) = \lim(y_n - x_n) = \lim(z_n) \overset{(21)}{\geq} 0,$$

implying

$$\lim(x_n) \leq \lim(y_n).$$

□

Theorem 67
If (x_n) is convergent and $a \leq x_n \leq b$ for (almost) all $n \in \mathbb{N}$, then

$$a \leq \lim(x_n) \leq b.$$

Proof
Defining $(z_n) = (a, a, a, \dots)$ and $(y_n) = (b, b, b, \dots)$ Theorem 66 implies

$$a = \lim(z_n) \leq \lim(x_n) \leq \lim(y_n) = b.$$

□

Theorem 68 (Squeeze/Sandwich Theorem 2)
Suppose that (x_n), (y_n) and (z_n) satisfy

$$x_n \leq y_n \leq z_n \tag{22}$$

for (almost) all $n \in \mathbb{N}$ and $\lim(x_n) = \lim(z_n)$. Then (y_n) is convergent and $\lim(x_n) = \lim(y_n) = \lim(z_n)$.

Proof
Let $\omega = \lim(x_n) = \lim(z_n)$. Given $\varepsilon > 0$ then there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$|x_n - \omega| < \varepsilon, \qquad |z_n - \omega| < \varepsilon.$$

By the assumption

$$x_n - \omega < y_n - \omega < z_n - \omega$$

for (almost) all $n \geq N$, implying

$$-\varepsilon < x_n - \omega < y_n - \omega < z_n - \omega < \varepsilon. \tag{23}$$

Let $\tilde{N} \in \mathbb{N}$ be such that (22) and (23) holds for all $n \geq \tilde{N}$. Then

$$|y_n - \omega| < \varepsilon$$

for all $n \geq \tilde{N}$, i.e.

$$\lim y_n = \omega.$$

□

Theorem 69
Let (x_n) converge to x. Then $(|x_n|)$ converges to $|x|$.

Sketch of Proof
By the reverse triangle inequality (Corollary 20)

$$\big||x_n| - |x|\big| \leq |x_n - x| \to 0.$$

□

Lecture 15 (February 06)

Theorem 70
Let (x_n) converge to x and $x_n \geq 0$ for all $n \in \mathbb{N}$. Then $(\sqrt{x_n})$ converges to $\sqrt{x}$.

Proof
We distinguish two cases.

- Assume $x = 0$. Given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$0 \leq x_n = |x_n - 0| < \varepsilon^2$$

for all $n \geq N$. Theorem 15 implies

$$0 \leq \sqrt{x_n} < \varepsilon$$

for $n \geq N$, i.e. $\lim(\sqrt{x_n}) = 0 = \sqrt{x}$.

- Assume $x > 0$. Then $\sqrt{x} > 0$ and

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{(\sqrt{x_n} + \sqrt{x})} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}.$$

Therefore

$$|\sqrt{x_n} - \sqrt{x}| = \frac{1}{\sqrt{x_n} + \sqrt{x}}|x_n - x| \leq \underbrace{\frac{1}{\sqrt{x}}}_{\text{const}} \underbrace{|x_n - x|}_{\to 0} \to 0.$$

□

Remark 71
One can show that if f is continuous (which hand-wavy means does not have jumps or diverges) in an open neighborhood of $x = \lim_{n\to\infty}(x_n)$, then $f(x) = \lim_{n\to\infty}(f(x_n))$. In fact this is a possible definition of continuous functions.

Theorem 72
Let (x_n) be such that $x_n > 0$ for all n and $L = \lim\left(\frac{x_{n+1}}{x_n}\right)$ exists. If $L < 1$ then $\lim(x_n) = 0$

In order to prove this we neet the following Lemma.

Lemma 73
Let $0 < b < 1$, then $\lim(b^n) = 0$

Sketch of Proof of Lemma 73 We did not define the logarithm, nor did we proof identities for calculating it. Also not that applying monotone functions to both sides does not change the inequality.

$$|b^n - 0| = b^n < \varepsilon \qquad \Leftrightarrow \qquad n \ln b < \ln \varepsilon \qquad \overset{\ln b<0}{\Leftrightarrow} \qquad n > \frac{\ln \varepsilon}{\ln b},$$

i.e. for $n \geq N \in \mathbb{N}$, where $N > \frac{\ln \varepsilon}{\ln b}$ one has $|b^n - 0| < \varepsilon$. □

Proof of Theorem 72
By Theorem 65 ($x_n \geq 0 \implies \lim(x_n) \geq 0$) $L \geq 0$. And by Theorem 40 there exists $r \in \mathbb{R}$ such that $0 \leq L < r < 1$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \underbrace{r - L}_{=\varepsilon},$$

implying

$$\frac{x_{n+1}}{x_n} < L + \varepsilon = L + (r - L) = r < 1.$$

Therefore

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_N r^{n-N+1}$$

Setting $C = \frac{x_N}{r^N}$, one has

$$0 < x_{n+1} < C r^{n+1}$$

for all $n \geq N$. Since $0 < r < 1$ one has $\lim(r^n) = 0$ by Lemma 73, which by the Sandwich Theorem (Theorem 60) implies

$$\lim(x_n) = 0.$$

□

Example 74
$(x_n) := \left(\frac{n}{2^n}\right) \to 0.$

Since $x_n > 0$ and

$$\frac{x_{n+1}}{x_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{1}{2}\underbrace{\left(1 + \frac{1}{n}\right)}_{\to 1} \to \frac{1}{2} < 1$$

Theorem 72 shows $\lim(x_n) = 0$.

Remark 75
Note that $x_n > 0$, $\frac{x_{n+1}}{x_n} < 1$ and $(L =) \lim\left(\frac{x_{n+1}}{x_n}\right) = 1$ might not be enough in order to get $(x_n) \to 0$ as the following counter example shows

Let $(x_n) = (\frac{n+1}{2n})$. Then

$$\frac{x_{n+1}}{x_n} = \frac{\frac{(n+1)+1}{2(n+1)}}{\frac{n+1}{2n}} = \frac{2n(n+2)}{2(n+1)^2} = \frac{2n^2+4n+2-2}{2(n+1)^2} = \frac{2(n^2+2n+1)-2}{2(n+1)^2}$$

$$= \frac{2(n+1)^2-2}{2(n+1)^2} = 1 - \frac{1}{(n+1)^2} \begin{cases} < 0 \text{ for all } n \in \mathbb{N} \\ \to 1 \end{cases}$$

but

$$(x_n) \to \frac{1}{2} \neq 0.$$

3.3 Monotone Sequences

Previously we had to know/guess the limit to show that a sequence is converging. Here we will learn some criteria when a sequence is convergent.

Definition 76
We say that (x_n) is

- increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ (strictly increasing if $x_n < x_{n+1}$)
- decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$ (strictly increasing if $x_n > x_{n+1}$)
- monotone if it is either increasing or decreasing.

Example 77

- (n) is (strictly) increasing
- $(\frac{1}{n})$ is (strictly) decreasing
- $(b^n)_{n\in\mathbb{N}}$ for $0 < b \in \mathbb{R}$ is monotone.

- $(1)_{n\in\mathbb{N}}$ is increasing, decreasing and monotone.

Theorem 78 (Monotone Convergence Theorem)
A monotone sequence (x_n) is convergent if and only if it is bounded.

1. If it is bounded and increasing
$$\lim(x_n) = \sup\{x_n \mid n \in \mathbb{N}\}$$
2. If it is bounded and decreasing
$$\lim(x_n) = \inf\{x_n \mid n \in \mathbb{N}\}$$

Lecture 16 (February 10)

Proof
" $\implies$ "
By Theorem 63 every convergent sequence is bounded.
" $\impliedby$ "
Assume that (x_n) is bounded. Since it is monotone it is increasing or decreasing.

1. Assume (x_n) is increasing. Since (x_n) is bounded the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded. Therefore (Completeness of $\mathbb{R}$) there exists $\tilde{x} = \sup\{x_n \mid n \in \mathbb{N}\}$. It is left to show that $\tilde{x} = \lim(x_n)$. Given $\varepsilon > 0$ by Lemma 28 there exits $x_k \in \{x_n \mid n \in \mathbb{N}\}$ such that
$$\tilde{x} - \varepsilon < x_k \overset{\text{increasing}}{\leq} x_n \leq \tilde{x} < \tilde{x} + \varepsilon$$
for all $n \geq k$, i.e.
$$|\tilde{x} - x_n| < \varepsilon$$
for all $n \geq k$, implying $\lim(x_n) = \tilde{x}$.
2. Assume (x_n) is decerasing. Let $(y_n) = (-x_n)$. Then y_n is increasing and bounded. Therefore Part 1 implies
$$\lim(y_n) = \sup\{y_n \mid n \in \mathbb{N}\} \overset{\text{Quiz 2}}{=} -\inf\{-y_n \mid n \in \mathbb{N}\} = -\inf\{x_n \mid n \in \mathbb{N}\}. \tag{24}$$
By Theorem 64.2 (Linearity of Limits)
$$\lim(x_n) = -\lim(y_n) \overset{(24)}{=} \inf\{x_n \mid n \in \mathbb{N}\}.$$

□

Lecture 17 (February 26)

Example 79 (The harmonic series)
Does $(h_n)_{n\in\mathbb{N}}$, where $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ converge?
It is monotone, since

$$h_{n+1} = \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}_{h_n} + \frac{1}{n+1} = h_n + \frac{1}{n+1} > h_n.$$

So by the Monotone Convergence Theorem (Theorem 78) it converges if and only if it is bounded. Next we'll show

$$h_{2^n} \geq 1 + \frac{n}{2}$$

for all $n \in \mathbb{N}$ by Induction.

$n = 1$

$$h_{2^n} = h_2 = 1 + \frac{1}{2} \geq 1 + \frac{n}{2}.$$

$n \to n+1$

$$\begin{aligned} h_{2^{n+1}} &= h_{2^n} + \underbrace{\frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}}}_{2^n \text{ terms, since } 2^{n+1}=22^n=2^n+2^n} \\ &\geq h_{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}} \\ &\geq h_{2^n} + \frac{2^n}{2^{n+1}} = h_{2^n} + \frac{1}{2} \geq 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}. \end{aligned}$$

Since $h_{2^n} \geq 1 + \frac{n}{2}$, which is unbounded, so also (h_{2^n}) is unbounded, implying that (h_n) are diverges.
This increases extremely slowly. In order to reach $h_n > 50$ one needs $\sim 10^{21}$ additions. A normal computer can do roughly $\sim 10^{12} - 10^{15}$ floating point operations per second.

Remark 80 (Limits of recursive sequences)
If one has proven that a recursive sequence (x_n) converges. Then one can find the limit if the limit commutes (is interchangeable) with the operations.
Example

Suppose one knows that

$$x_1 = 2, \qquad x_{n+1} = 2 + \frac{1}{x_n}$$

converges. Then since $0 < 2 \le x_n$ for all n

$$x = \lim(x_{n+1}) = \lim\left(2 + \frac{1}{x_n}\right) \overset{\text{Theorem 64}}{=} 2 + \lim\left(\frac{1}{x_n}\right)$$
$$\overset{\text{Theorem 64},\ 0<2<x_n}{=} 2 + \frac{1}{\lim(x_n)} = 2 + \frac{1}{x}$$

holds, implying

$$x^2 - 2x - 1 = 0 \implies x = 1 \pm \sqrt{2} \implies x = 1 + \sqrt{2}$$

It is important to check that the sequence converges!
Counterexample

$$x_1 = 1, \qquad x_{n+1} = 2x_n + 1$$

Simply calculating yields

$$x = \lim(x_{n+1}) = 2\lim(x_n) + 1 = 2x + 1$$
$$\implies x = -1,$$

but x_n is positive for all n.

Euler's Number

Proposition 81 (Binomial Theorem)
Let $n \in \mathbb{N}_0$ and $x, y \in \mathbb{R}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Notation
Here $\sum$ is notation for repeated addition, defined by

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

for $m, n \in \mathbb{Z}$ with $m \ge n$ and $a_k \in \mathbb{R}$ for $k = m, \ldots, n$. The binomial coefficient is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

for $n, k \in \mathbb{N}_0$ with $n \ge k$, where ! is the factorial, which is recursively defined by

$$j! = j\,(j-1)!, \qquad 0! = 1$$

for $j \in \mathbb{N}_0$.

The Binomial Theorem can be proved using induction and Pascal's rule. You can find a fundamental proof of both here.

Example 82 (Euler's number)
Let

$$e_n = \left(1 + \frac{1}{n}\right)^n.$$

Then

- (e_n) is increasing

  Proof
  By the Binomial Theorem

$$\begin{aligned}
e_n &= \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{1}{n^k} \\
&= \frac{n!}{0!(n-0)!}\frac{1}{n^0} + \frac{n!}{1!(n-1)!}\frac{1}{n^1} + \frac{n!}{2!(n-2)!}\frac{1}{n^2} + \frac{n!}{3!(n-3)!}\frac{1}{n^3} \\
&\quad + \cdots + \frac{n!}{n!0!}\frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n^2} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n^3} \\
&\quad + \cdots + \frac{1}{n!}\frac{n(n-1)(n-2)\ldots 3\cdot 2\cdot 1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \\
&\quad + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)
\end{aligned}$$

  and therefore

$$\begin{aligned}
e_{n+1} &= 1 + 1 + \frac{1}{2!}\underbrace{\left(1 - \overbrace{\frac{1}{n+1}}^{\leq \frac{1}{n}}\right)}_{\geq \left(1-\frac{1}{n}\right)} + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) \\
&\quad + \cdots + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right) \\
&\quad + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right) \\
&\geq e_n + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right) \\
&\geq e_n
\end{aligned}$$

  □

Lecture 18 (February 27)

- (e_n) is bounded

  Proof
  Quiz ☐

- Since it is bounded and increasing it converges to some number e, Euler's number. Calculating one finds

n	e_n
10	2.59374
100	2.70481
1000	2.71692
10000	2.71814
100000	2.71826
1000000	2.71828

3.4 Subsequences and Bolzano-Weierstrass

Definition 83
Given $(x_n)_{n\in\mathbb{N}}$ and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers, the sequence $(x_{n_k})_{k\in\mathbb{N}}$, i.e.

$$(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$$

is called a subsequence of (x_n).

Example 84
For

$$(x_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots\right)$$

a subsequence is given by

$$(x_{n_k})_{k\in\mathbb{N}} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right) = (x_{2k})_{k\in\mathbb{N}}$$

where $(n_k)_{k\in\mathbb{N}} = (2k) = (2, 4, 6, \dots)$

Theorem 85
If $(x_n)_{n\in\mathbb{N}}$ converges to $x \in \mathbb{R}$, then any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ also converges to x.

Proof
We first show that $n_k \geq k$ for all $k \in \mathbb{N}$ by induction.

- $k = 1$: $n_k \in \mathbb{N}$ by definition, implying $n_k \geq 1$.
- $k \to k+1$: Since $n_k \in \mathbb{N}$ for all k and $(n_k)_{k\in\mathbb{N}}$ is strictly increasing (i.e. $n_{k+1} > n_k$) one has $n_{k+1} \geq n_k + 1 \overset{\text{induction hypothesis}}{\geq} k + 1$.

By definition for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$|x_k - x| < \varepsilon,$$

and therefore for $n_k \geq k \geq N$

$$|x_{n_k} - x| < \varepsilon.$$

□

Theorem 86
Given (x_n), the following are equivalent.

1. (x_n) does not converge to $x \in \mathbb{R}$
2. There exists an $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon$
3. There exists an $\varepsilon > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x| \geq \varepsilon$ for all $k \in \mathbb{N}$.

Proof
1 $\Longrightarrow$ 2
If (x_n) does not converge to x, then there exits an $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exits $n(N) \geq N$ such that

$$|x_n - x| \geq \varepsilon.$$

Renaming $N \to k$, for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ with $n_k \geq k$ such that

$$|x_{n_k} - x| \geq \varepsilon. \tag{25}$$

2 $\Longrightarrow$ 3
We have x_{n_k} which fulfill everything except that they are a subsequence. So it is left to show that we can find n_k such that $n_{k+1} > n_k$. By 2 with $k = 1$ there exists $\tilde{n}_1 \geq 1$ satisfying (25). By 2 with $k = \tilde{n}_i + 1$ there exists $\tilde{n}_{\tilde{n}_i+1} \geq \tilde{n}_i + 1 > \tilde{n}_i$ satisfying (25), i.e. these $\tilde{n}$ build a an increasing sequence of natural numbers. Therefore, renaming $n_k = \tilde{n}_{\tilde{n}_k+1}$ yields the claim.
3 $\Longrightarrow$ 2
3 says that (x_{n_k}) can not converge to x. Therefore Theorem 85 yields that (x_n) can not converge to x. □

Corollary 87 (Divergence Criteria)
If (x_n) has either of the following properties, then it diverges.

1. It has 2 subsequences (x_{n_k}) and (x_{r_k}), whose limits are not the same
2. It is unbounded.

Proof
1 follows by Theorem 86. 1 follows by Theorem 63 (Convergent sequences are bounded). □

Example 88
For $(x_n) = ((-1)^n)$

- $(x_{2n}) = (1)$ converges to 1
- $(x_{2n+1}) = (-1)$ converges to -1.

Theorem 89
Every sequence of real numbers has a monotone subsequence.

Proof
For the purpose of this proof call x_m *peak* if $x_m \geq x_n$ for all $n \geq m$. x_m is never exceeded by any term that follows. If

- there exist finitely many (potentially 0) peaks $x_{m_1}, \dots x_{m_r}$ listed by in increasing subscripts, then for $s_1 = m_r + 1$ x_{s_1} is not a peak, implying there exists $x_{s_2} > x_{s_1}$, which again is not a peak, implying there exists $x_{s_3} > x_{s_2}$. Inductively we find an an increasing sequence (x_{s_k}).
- there exist infinitely many peaks $x_{m_1}, x_{m_2}, \dots$ (again $m_1 < m_2 < \dots$), i.e. there exits a subsequence of peaks $(x_{m_k})_{k \in \mathbb{N}}$. Then since each term is a peak

  $$x_{m_1} \geq x_{m_2} \geq \dots \geq x_{m_k} \geq \dots$$

  this subsequence is decreasing.

□

Lecture 19 (March 03)

Theorem 90 (Bolzano-Weierstraß)
Every bounded sequence of real numbers has a convergent subsequence.

Proof
By Theorem 89 the sequence has a monotone subsequence which is again bounded. By the Monotone Convergence Theorem 78 (which says that every bounded monotone sequence is convergent) this subsequence converges. □

Remarks 91

- There might be multiple convergent subsequences (e.g. $(x_n) = ((-1)^n)$)
- Generalizations of this theorem are used in many aspects of mathematics
- It does also hold in finitely many dimensions, i.e. for $(x_n)_{n\in\mathbb{N}}$, where $x_n \in \mathbb{R}^k$ are k-dimensional vectors. The proof works by consecutively taking subsequences, where in the n-th subsequence leads to the convergence of the n-th component.
- It does not hold in infinite dimensional spaces, but a weaker results holds. For example in a Hilbert space every bounded sequence has a weakly convergent subsequence. Example for a sequence that does not converge in an infinite dimensional space:

  Space: $L^2(0,1)$ is the space of functions which square is integrable on $(0,1)$, equipped with the norm $||f||_{L^2} = \sqrt{\int_0^1 f^2(x)\ dx}$

  (The corresponding objects in our lecture are the space $\mathbb{R}$ and the norm $\|x\|_{\mathbb{R}} = |x|$)

  Sequence: $f_n(x) = \sin(2\pi nx)$

  Boundedness:

  $$\|f_n\|_{L^2} = \|\sin(2\pi nx)\|_{L^2} = \sqrt{\int_0^1 \sin^2(2\pi nx)\ dx} = \overset{\text{calculation}}{\cdots} = \sqrt{\frac{1}{2}}$$

  Non-Convergence: Loosely speaking, as n increases f_n oscillates with higher and higher frequencies, so to which value should it converge? For every ε-neighborhood U_ε of every number in $(0,1)$ there exists $N \in \mathbb{N}$ such that for $n \geq N$ there exists $x_+, x_- \in U_\varepsilon$ such that $f_n(x_+) > \frac{1}{2}$, $f_n(x_-) < -\frac{1}{2}$, but $f_{n+1}(x_+) < -\frac{1}{2}$, $f_{n+1}(x_-) > \frac{1}{2}$. Mathematically, it converges weakly to 0.

Theorem 92
Let (x_n) be bounded and assume that every convergent subsequence of (x_n) converges to x. Then (x_n) converges to x.

Proof
By Contradiction, i.e. assume (x_n) does not converge to x. Then by Theorem 86 there exists $\varepsilon > 0$ and a subsequence (x_{n_k}) such that

$$|x_{n_k} - x| \geq \varepsilon \tag{26}$$

for all $k \in \mathbb{N}$. Since (x_n) is bounded, there exists $M > 0$ such that $|x_n| \leq M$, implying $|x_{n_k}| \leq M$, i.e. (x_{n_k}) is bounded. By Bolzano-Weierstraß (Theorem 90) there exists a convergent subsequence $(x_{n_{k_l}})$, which is also a subsequence of

(x_n) and according to the hypothesis has to converge to x. Thus there exists $N \in \mathbb{N}$ such that

$$|x_{n_{k_l}} - x| < \varepsilon$$

for all $l \geq N$, a contradiction to (26). □

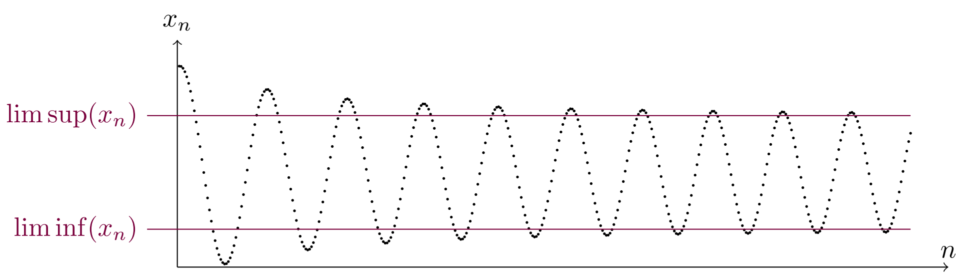
Definition 93
Let (x_n) be bounded.

- The limit superior, $\limsup(x_n)$, or $\overline{\lim}(x_n)$, is defined by

$$\limsup(x_n) = \inf \{v \in \mathbb{R} \,|\, v < x_n \text{ for at most finitely many } n\}$$

- The limit inferior, $\liminf(x_n)$ or $\underline{\lim}(x_n)$, is defined by

$$\liminf(x_n) = \sup \{v \in \mathbb{R} \,|\, v > x_n \text{ for at most finitely many } n\}$$



3.5 Cauchy Sequences

Definition 94
A sequence (x_n) is called Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq N$ it holds

$$|x_n - x_m| < \varepsilon$$

Example 95
$(\frac{1}{n})$ is a Cauchy sequence

Given $\varepsilon > 0$, let $N > \frac{2}{\varepsilon}$. Then for $m, n \geq N$

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{m} + \frac{1}{n} \leq \frac{1}{N} + \frac{1}{N} < \varepsilon.$$

Theorem 96
A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof
" $\Longrightarrow$ "
If (x_n) converges to x, then for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n \geq \mathbb{N}$

$$|x_m - x| < \frac{\varepsilon}{2}, \qquad |x_n - x| < \frac{\varepsilon}{2}$$

Therefore, by triangle inequality

$$|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| < \varepsilon.$$

" $\Longleftarrow$ " Let (x_n) be a Cauchy sequence. We will show that it is bounded, therefore Bolzano-Weierstrass yields a convergent subsequence. Then we can estimate all terms with respect to that limit.

- (boundedness) By assumption there exits an $N \in \mathbb{N}$ such that for all $m, n \geq N$

$$|x_m - x_n| \leq 1 \qquad \Longrightarrow \qquad |x_n - x_N| \leq 1$$

  Therefore for all $n \geq N$

$$|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| \leq |x_n - x_N| + 1.$$

  Defining

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

  yields an upper bound for all x_n for $n \in \mathbb{N}$, i.e. (x_n) is a bounded sequence

- (existence of convergent subsequence) By Bolzano-Weierstraß (Theorem 90), since (x_n) is a bounded sequence, there exists a convergent subsequence, i.e.

$$x = \lim_{k \to \infty} (x_{n_k}).$$

- (conclusion) Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence there exits $N \in \mathbb{N}$

$$|x_n - x_m| < \frac{\varepsilon}{2} \tag{27}$$

  for all $m, n \geq N$ and since (x_{n_k}) converges to x, there exists $K \in \mathbb{N}$ such that

$$|x_{n_k} - x| < \frac{\varepsilon}{2} \tag{28}$$

  for all $k \geq K$. Choosing a suffieciently big k, it holds $n_k \geq N$, implying that (27) holds with $m = n_k$. Then triangle inequality yields

$$|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| \overset{(27),(28)}{<} \varepsilon$$

  for all $n \geq N$.

□

Lecture 20 (March 05)

Remarks 97

- Cauchy sequences are important in more general spaces
- While the result seems obvious it only holds in complete spaces. $\mathbb{Q}$ is not complete. It can be shown that the sequence

$$x_1 = 1, \qquad x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2}$$

  satisfies

  - $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$, i.e. it is a sequence in $\mathbb{Q}$
  - $\lim(x_n) = \sqrt{2} \notin \mathbb{R}$

- (In more general mathematics, the concept of a Cauchy sequence is more general then one of a converging sequence. Meaning every converging sequence is a Cauchy sequence but not necessary every Cauchy sequence is a converging sequence (as the previous counterexample indicates).)

Definition 98
(x_n) is a contractive sequence if there exists a constant $0 < C < 1$ such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

for all $n \in \mathbb{N}$.

Theorem 99
Every contractive sequence is a Cauchy sequence and convergent.

Proof
Successively applying the condition

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \leq C^2|x_n - x_{n-1}| \leq \cdots \leq C^n|x_2 - x_1|. \quad (29)$$

By Quiz 3.1.2

$$\sum_{j=0}^{n} r^j := r^0 + r^1 + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad (30)$$

for all $n \in \mathbb{N}$ and $1 \neq r \in \mathbb{R}$. For $m \geq n$ repeated application of the triangle inequality yields

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - \ldots - x_{n+1} + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\overset{(29)}{\leq} \left(C^{m-2} + C^{m-3} + \cdots + C^{n-1}\right) |x_2 - x_1| \\ &= C^{n-1} \left(C^{m-n-1} + C^{m-n-2} + \cdots + C^0\right) |x_2 - x_1| \\ &\overset{(30)}{=} C^{n-1} \frac{1 - C^{m-n}}{1 - C} |x_2 - x_1| \\ &\leq C^n \underbrace{\frac{1}{C - C^2} |x_2 - x_1|}_{\leq \text{const}} \end{aligned}$$

and since by Lemma 73 $\lim_{n\to\infty} C^n = 0$, $|x_m - x_n| < \varepsilon$ for sufficiently large $m \geq n \in \mathbb{N}$, implying it is a Cauchy sequence implying it converges by Theorem 96. □

3.6 Properly Diverging Sequences

Definition 100
We say a sequence (x_n)

- tends to ∞ and write $\lim_{n\to\infty}(x_n) = \infty$ if for every $r \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ it holds $x_n > r$.
- tends to $-\infty$ and write $\lim_{n\to\infty}(x_n) = -\infty$ if for every $r \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ it holds $x_n < r$.
- is properly divergent if $\lim_{n\to\infty}(x_n) = \infty$ or $\lim_{n\to\infty}(x_n) = -\infty$

Remark 101
Here (as always) ∞ is just notation.

Example 102

- $\lim_{n\to\infty}(n) = \infty$.

  By the Archimedian Property for any $r \in \mathbb{R}$ there exists $N \in \mathbb{N}$ with $N > r$ and therefore $n \geq N > r$ for $n \geq N$.

Theorem 103 (Comparison Principle)
Given $C > 0$, let (x_n) and (y_n) satisfy

$$x_n \leq C y_n \tag{31}$$

for (almost) all $n \in \mathbb{N}$. If

- $\lim(x_n) = \infty$, then $\lim(y_n) = \infty$.

- $\lim(y_n) = -\infty$, then $\lim(x_n) = -\infty$.

Proof
Assume $\lim(x_n) = \infty$. Then for every $\tilde{r} = rC \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that

$$rC = \tilde{r} < x_n \overset{(31)}{\leq} Cy_n \quad \Longrightarrow \quad r < y_n$$

for (almost) all $n \geq N$, implying $\lim(y_n) = \infty$. The other implication follows analogously. □

Theorem 104 (Another Comparison Principle)
Let (x_n) and (y_n) satisfy

$$x_n, y_n > 0$$

for all $n \in \mathbb{N}$ and

$$\lim\left(\frac{x_n}{y_n}\right) = L \tag{32}$$

for some $L > 0$. Then $\lim(x_n) = \infty$ if and only if $\lim(y_n) = \infty$.

Proof
By (32) there exists $N \in \mathbb{N}$ such that

$$\left|\frac{x_n}{y_n} - L\right| < \frac{L}{2} \qquad \Longrightarrow \qquad \frac{1}{2}L < \frac{x_n}{y_n} < \frac{3}{2}L$$

for all $n \geq N$, implying

$$y_n < \frac{2}{L}x_n, \qquad x_n < \frac{3}{2}Ly_n.$$

Therefore using Theorem 103 twice, with $C = \frac{2}{L}$ and $C = \frac{3}{2}L$, yields the claim. □

Lecture 21 (March 06)

3.7 Infinite Series

Definition 105
Given a sequence (x_n), the sequence (s_k), defined by

$$s_k = \sum_{n=1}^{k} x_n$$

or equivalently

$$
\begin{aligned}
s_1 &= x_1 & &= x_1 \\
s_2 &= s_1 + x_2 & &= x_1 + x_2 \\
&\vdots & &\vdots \\
s_k &= s_{k-1} + x_k & &= x_1 + x_2 + \cdots + x_k
\end{aligned}
$$

is called the (infinite) series generated by (x_n). If $\lim(s_k)$ exists, we say the series is convergent and call the limit the sum or value of the series. If the series does not converge it is called divergent. The x_n are called the terms of the series and the s_k are called the partial sums.
The series, but also its limit is denoted by

$$
\sum(x_n), \qquad \sum x_n, \qquad \sum_{n=1}^{\infty} x_n
$$

and both are sometimes called (infinite) series.

Remarks 106

- There is some ambiguity in this definition.
- The ∞ in $\sum_{n=1}^{\infty} x_n$ is just notation. It just represents that the sequence (s_k) defined by $s_k = \sum_{n=1}^{k} x_k$ converges in the sense of ε-convergence. There exists $s \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $k \geq N$

$$
\left| \sum_{n=1}^{k} x_n - s \right| = |s_k - s| < \varepsilon.
$$

- The series might be indexed starting from another value

$$
\sum_{n=0}^{\infty} x_n \quad \text{or} \quad \sum_{n=-82}^{\infty} x_n
$$

- The usual algebraic rules for sums do not work for infinite series, $\sum_{n=1}^{\infty} x_n$. Consider $\sum_{k=0}^{\infty}(-1)^k$.
  - Associativity

$$
\begin{aligned}
\text{"} \sum_{k=0}^{\infty}(-1)^k &= \underbrace{1-1}_{=0} + \underbrace{1-1}_{=0} + \underbrace{1-1}_{=0} + \underbrace{1-1}_{=0} + \dots \\
&= 0 + 0 + 0 + \cdots = 0\text{"}
\end{aligned}
$$

but

$$\text{"}\sum_{k=0}^{\infty}(-1)^k = 1 - \underbrace{(1-1)}_{=0} - \underbrace{(1-1)}_{=0} - \underbrace{(1-1)}_{=0} - \underbrace{(1-1)}_{=0} + \dots$$
$$= 1\text{"}$$

– Commutativity

$$\begin{aligned}\text{"}\sum_{k=0}^{\infty}(-1)^k &= 1-1+1-1+1+\dots \\ &= 1+1-1+1+1-1+1+1-1+\dots \\ &= 2\sum_{k=0}^{\infty}1 - \sum_{k=0}^{\infty}1 = \sum_{k=0}^{\infty}1 = \infty\text{"}\end{aligned} \tag{33}$$

but

$$\begin{aligned}\text{"}\sum_{k=0}^{\infty}(-1)^k &= 1-1+1-1+1+\dots \\ &= 1-1-1+1-1-1+1-1-1+\dots \\ &= \sum_{k=0}^{\infty}1 - 2\sum_{k=0}^{\infty}1 = -\sum_{k=0}^{\infty}1 = -\infty\text{"}\end{aligned}$$

Showing, with similar tricks, $\sum_{k=0}^{\infty} k = -\frac{1}{12}$ is a math meme, (although one can argue why assigning $-\frac{1}{12}$ to this series is not completely crazy)

- Determine the value by finding the limit of its partial sum instead of calculating the infinite sum!

Example 107

1. Geometric Series

   Given $r \in \mathbb{R}$ consider

$$\sum_{k=0}^{\infty} r^k.$$

   We will show that it converges for $|r| < 1$ and

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{for } |r| < 1.$$

   - Either by Quiz 3.1.2

$$s_k = \sum_{n=0}^{k} r^n = \frac{1-r^{k+1}}{1-r}$$

- or explicitly the corresponding sequence is given by

$$s_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + \cdots + r^k$$

and

$$\begin{aligned} s_k(1-r) =& 1 + r + r^2 + \cdots + r^k \\ & \quad r - r^2 - r^3 - \cdots - r^{k+1} \\ =& 1 - r^{k+1}, \end{aligned}$$

implying

$$s_k = \frac{1 - r^{k+1}}{1 - r}.$$

This suggest that $s_k \to \frac{1}{1-r}$ Therefore

$$\left| s_k - \frac{1}{1-r} \right| = \left| \frac{1 - r^{k+1} - 1}{1-r} \right| \leq \underbrace{\frac{1}{|1-r|}}_{\leq \text{const}} \underbrace{|r^{k+1}|}_{\to 0 \text{ by Lemma 73}} \leq C|r^{k+1}|$$

for some $0 < C \in \mathbb{R}$, implying $\lim_{k\to\infty}(s_k) = \frac{1}{1-r}$ by the sandwich theorem (60) and Lemma 73.

2. For $r = -1$ the geometric series is given by

$$\sum_{k=0}^{\infty} (-1)^k,$$

which with sequences

$$s_k = \underbrace{1 + (-1) + (+1) + (-1) + (+1) + (-1) + \ldots}_{k \text{ terms}},$$

i.e. $(s_k) = (1, 0, 1, 0, 1, \ldots)$ which does not converge (analogous to $(1, -1, 1, -1, \ldots)$ not converging), implying $\sum_{k=0}^{\infty}(-1)^k$ is divergent.

3. Euler's number satisfies (and can be equivalently defined by its series)

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}$$

4. The exponential function, which can be defined by,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for $x \in \mathbb{R}$.

5. The natural logarithm, which can be defined by,

$$\ln x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$$

converges for $x > 0$

For $x = 0$ this corresponds to the negative of the harmonic series (Example 79), which diverges.

6. Sine/Cosine, defined by,

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \qquad \cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k},$$

converge for any $x \in \mathbb{R}$.

Lecture 22 (March 10)

Theorem 108
If $\sum x_n$ converges, then $\lim(x_n) = 0$

Proof
The partial sum satisfies

$$s_n = s_{n-1} + x_n$$

and therefore

$$x_n = s_n - s_{n-1}.$$

Since (s_k) converges by assumption, the linearity of limits implies

$$\lim(x_n) = \lim(s_n - s_{n-1}) = \lim(s_n) - \lim(s_{n-1}) = 0.$$

□

Corollary 109
The series $\sum x_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m > n \geq N$, then

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon.$$

This follows immediately from Theorem 96 (Real sequences converge if and only if they are Cauchy sequences).

Theorem 110
Let (x_n) be a sequence of nonnegative real numbers. Then $\sum x_k$ converges if and only if (s_k) is bounded. In that case

$$\sum_{k=1}^{\infty} x_k = \lim(s_k) = \sup\{s_k \mid k \in \mathbb{N}\}$$

Proof
Since $x_n \geq 0$ the sequence of partial sums is monotone increasing and by the monotone convergence theorem (78) the it converges if and only if it is bounded, in which case its limit is equal to $\sup\{s_k\}$. □

Theorem 111 (Comparison of Series)
Suppose that for (x_n) and (y_n) there exists $N \in \mathbb{N}$ such that

$$0 \leq x_n \leq y_n \tag{34}$$

for all $n \geq N$.

1. If $\sum y_n$ converges, then $\sum x_n$ converges
2. If $\sum x_n$ diverges, then $\sum y_n$ diverges

Proof

1. Suppose $\sum y_n$ converges, then there exists $K > 0$ that for all $m > n \geq N$

   $$\sum_{n=N}^{m} y_n < K$$

   holds. By (34) also

   $$\sum_{n=N}^{m} x_n \leq \sum_{n=N}^{m} y_n < K$$

   for all $m > n \geq N$, implying that the tail starting at N is a bounded increasing series and by theorem 110 convergent, implying that $\sum_{N=1}^{\infty} x_n$ is converging.

2. This is the contrapositive of 1.

□

Theorem 112
Suppose (x_n) and (y_n) are strictly positive sequences and suppose that the limit

$$r = \lim\left(\frac{x_n}{y_n}\right)$$

exists.

1. If $r \neq 0$, then $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.
2. If $r = 0$ and $\sum y_n$ is convergent, then $\sum x_n$ is convergent.

Proof

1. Setting $\varepsilon = \frac{r}{2}$, by the definition of the limit there exists $N \in \mathbb{N}$ such that

   $$\left|\frac{x_n}{y_n} - r\right| < \frac{r}{2},$$

   implying

   $$-\frac{r}{2} < \frac{x_n}{y_n} - r < \frac{r}{2} \implies \frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2}.$$

   Therefore

   $$\frac{1}{2} r y_n \leq x_n \leq \frac{3}{2} r y_n$$

   and applying the Comparison theorem 111 with $\tilde{x}_n = \frac{1}{2} r y_n$ and $\tilde{y}_n = x_n$ and again with $\tilde{x}_n = x_n$ and $\tilde{y}_n \frac{3}{2} r y_n$ yields claim 1.

2. Similar if $r = 0$, then setting $\varepsilon = 1$ yields

   $$\left|\frac{x_n}{y_n} - 0\right| < 1 \implies 0 < x_n \leq y_n$$

   for $n \geq N$ and the Comparison theorem 111 yields the claim.

□

Corollary 113 (Linearity of Convergent Series)
Assume that $\sum x_n$ and $\sum y_n$ converge and $c \in \mathbb{R}$. Then

$$\sum (x_n + c y_n) = \sum x_n + c \sum y_n.$$

This follows immediately from the linearity of limits (Theorem 64)

Remarks 114

- The assumption that $\sum x_n$ and $\sum y_n$ are convergent is crucial, which can be seen by (33).
- Multiplication, called the Cauchy Product is a little bit more complicated because of mixed terms

Lecture 23 (March 12)

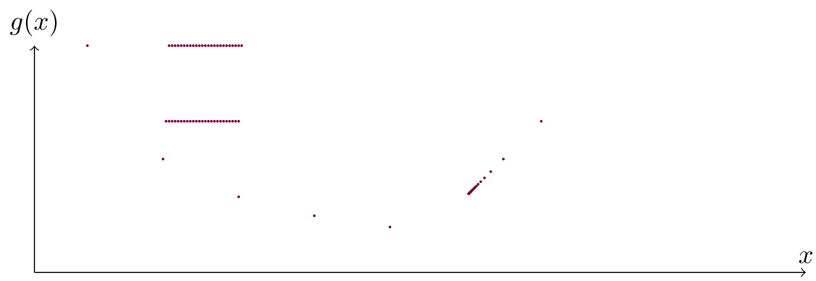
4 Limits

4.1 Limits of Functions

While many people think of functions as something like $f(x) = x^2$, where it is quite easy to define limits, in general functions can "ugly" as

$$g(x) = \begin{cases} \frac{3}{x} & \text{for } x \in \mathbb{N} \text{ and } x < 5 \\ 2 & \text{for } x \in \mathbb{Q} \text{ and } 2 < x < 3 \\ 3 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \text{ and } 2 < x < 3 \\ x - 5 & \text{for } x \in \left\{ 6 + \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} \text{ and } 5 < x \end{cases} \tag{35}$$

What is $\lim_{x \to 1} g(x)$? Is $\lim_{x \to 6} g(x) = 1$? What is $\lim_{x \to \frac{5}{2}} g(x)$?



The function defined in (35).

Definition 115
Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is called cluster point of A if for every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$

Roughly speaking a cluster point is if there are (infinitely) many points in A that are arbitrary close to x
c might be in A or not, it does not matter.

Theorem 116
$c \in \mathbb{R}$ is a cluster point of $A \subset \mathbb{R}$ if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

Proof
" $\implies$ "
If c is a cluster point then for any $n \in \mathbb{N}$ the $\frac{1}{n}$-neighborhood $V_{\frac{1}{n}}(c)$ contains at least one point $a_n \in A$ with $a_n \neq c$. Therefore $a_n \in A$, $a_n \neq c$ and $|a_n - c| < \frac{1}{n} \implies \lim(a_n) = c$.
" $\impliedby$ "
If there exists a sequence (a_n) in $A \setminus \{c\}$ with $\lim(a_n) = c$, then for any $\delta > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - c| < \delta$, i.e. there exists $a_n \in A$ and $a_n \neq c$ such that $|a_n - c| < \delta$. □

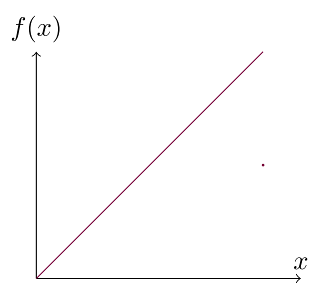
Example 117

- For $A_1 = (0,1)$ every point in $[0,1]$ is a cluster point of A_1
- For $\mathbb{R}$ every point $c \in \mathbb{R}$ is a cluster point of $\mathbb{R}$
- A finite set has no cluster points
- $\mathbb{N}$ has no cluster points
- $A_2 = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$ has only the cluster point 0
- For $A_3 = \mathbb{Q} \cap [0,1]$ every point $x \in [0,1]$ is a cluster point of A_3

Definition 118
Let $A \subset \mathbb{R}$ and c be a cluster point of A. For a function $f : A \to \mathbb{R}$ a number $L \in \mathbb{R}$ is said to be a limit of f at c if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$ it holds $|f(x) - L| < \varepsilon$.
If this limit exists we write $L = \lim_{x \to c} f(x)$ and say $f(x)$ converges to L at c or $f(x) \to L$ as $x \to c$ and say $f(x)$ approaches L as x approaches c.
If no such limit exists we say f diverges at c.

Remarks 119

- Since δ might depend on ε and we will sometimes write $\delta(\varepsilon)$
- $0 < |x - c|$ implies that only $x \neq c$ are relevant in the definition. In particular for $f(x) = \begin{cases} x & \text{for } x < 1 \\ \frac{1}{2} & \text{for } x = 1 \end{cases}$ one has $\lim_{x \to 1} f(x) = 1 \neq \frac{1}{2} = f(1)$



- Convergence (in this sense) is a local property, i.e. it is only important what happens for $x \in V_\delta(c)$ with small δ. draw picture

Theorem 120 (Uniqueness)
These limits are unique.

Proof
Suppose L, L' satisfy the definition. Then for any $\varepsilon > 0$ there exists $\delta^\star(\frac{\varepsilon}{2}), \delta'(\frac{\varepsilon}{2}) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta^\star(\frac{\varepsilon}{2})$, then $|f(x) - L| < \frac{\varepsilon}{2}$ and if $x \in A$ and $0 < |x - c| < \delta'(\frac{\varepsilon}{2})$, then $|f(x) - L'| < \frac{\varepsilon}{2}$. Setting $\delta = \min\{\delta^\star(\frac{\varepsilon}{2}), \delta'(\frac{\varepsilon}{2})\}$. Then if $x \in A$ and $0 < |x - c| < \delta$

$$0 \leq |L - L'| = |L - f(x) + f(x) - L'| \leq |L - f(x)| + |f(x) - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

and since $\varepsilon > 0$ is arbitrary $L' = L$. □

Theorem 121
Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent

1. $\lim_{x\to c} f(x) = L$
2. For any ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ of c such that for any $x \neq c$ in $V_\delta(c) \cap A$, $f(x) \in V_\varepsilon(L)$

Proof
This follows directly by observing that $V_\delta(c) = (c-\delta, c+\delta) = \{x \in \mathbb{R} \,|\, |x - c| < \delta\}$ and $V_\varepsilon(L) = (L - \varepsilon, L + \varepsilon) = \{y \in \mathbb{R} \,|\, |y - L| < \varepsilon\}$. □

Example 122

- $\lim_{x\to c} b = b$, i.e. for $b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}, f(x) = b$ we show that $\lim_{x\to c} f(x) = b$.

  Let $\varepsilon > 0$. Then for $\delta = 1$ if $0 < |x - c| < 1$, we have

  $$|f(x) - b| = |b - b| = 0 < \varepsilon.$$

- $\lim_{x\to c} x = c$, i.e. for $f : \mathbb{R} \to \mathbb{R}, f(x) = x$ we show that $\lim_{x\to c} f(x) = c$

  Let $\varepsilon > 0$. Then for $\delta(\varepsilon) = \varepsilon$ if $0 < |x - c| < \delta = \varepsilon$ one has $|f(x) - c| = |x - c| < \varepsilon$.

- $\lim_{x\to c} x^2 = c^2$, i.e. for $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ we show that $\lim_{x\to c} f(x) = c^2$.

  Calculating

  $$|f(x) - c^2| = |x^2 - c^2| = |(x + c)(x - c)| = |x + c|\,|x - c|$$

  and if $|x - c| < 1$, then

  $$|x + c| \leq |x| + |c| \leq |x - c + c| + |c| \leq |x - c| + 2|c| < 1 + 2|c|$$

  implying

  $$|f(x) - c^2| = |x + c|\,|x - c| < (2|c| + 1)|x - c| \qquad (36)$$

We want the right hand-side $< \varepsilon$, so we set

$$(2|c|+1)|x-c| \overset{!}{<} \varepsilon \Leftrightarrow |x-c| < \underbrace{\frac{\varepsilon}{2|c|+1}}_{\overset{!}{>}\delta}$$

Therefore given $\varepsilon > 0$ let $\delta(\varepsilon) = \min\{1, \frac{\varepsilon}{2|c|+1}\}$. Then for $x \in \mathbb{R}$ with $0 < |x-c| < \delta(\varepsilon)$

$$|f(x) - c^2| \overset{(36)}{<} (2|c|+1)|x-c| < (2|c|+1)\delta \leq \varepsilon$$

- $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ if $c > 0$.

  Do it as an exercise. If you are stuck take a look at Theorem 64.5. If you are still stuck look at Example 4.1.7d in the book.

- $\lim_{x \to 2} \frac{x^3-4}{x^2+1} = \frac{4}{5}$

  Do it as an exercise.

End of material for Test 2.

Theorem 123 (Sequential Criterion)
Let $f : A \to \mathbb{R}$ and c be a cluster point of A. Then the following are equivalent.

1. $\lim_{x \to c} f(x) = L$
2. For every sequence (x_n) in A that converges to c with $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

Proof
1. $\implies$ 2.
Let ε. Then there exists $\delta > 0$ such that if $x \in A$ satisfies

$$0 < |x-c| < \delta$$

then

$$|f(x) - L| < \varepsilon. \tag{37}$$

Given a sequence (x_n) in A that converges to c and $x_n \neq c$ for all $n \in \mathbb{N}$, then by definition for every $\delta > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$0 \overset{x_n \neq c}{<} |x_n - c| < \delta,$$

which by (37) implies

$$|f(x_n) - L| < \varepsilon,$$

i.e. $(f(x_n))$ converges to L.

2. $\implies$ 1. by contrapositive
If 1 is not true, then there exists an ε-neighborhood $V_\varepsilon(L)$ of L such for every δ-neighborhood of c there exists at least one $x_\delta \in A \cap V_\delta(c)$ with $x_\delta \neq c$ such that $f(x_\delta) \notin V_\varepsilon(L)$. Therefore for every $n \in \mathbb{N}$, the $\frac{1}{n}$-neighborhood of c contains at least one $x_n \in A$ such that

$$0 < |x_n - c| < \frac{1}{n}$$

but

$$|f(x_n) - L| \geq \varepsilon$$

for all $n \in \mathbb{N}$. This implies that the sequence (x_n) is in A, $x_n \neq c$ for all $n \in \mathbb{N}$ and $(f(x_n))$ does not converge to L. So we have shown that not 1 implies not 2. □

This shows that many properties from sequences carry over to limits, and both concepts are similar

Lecture 24 (March 17)

Corollary 124 (Divergence Criteria)
Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A.

1. f does not converge to L at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that (x_n) converges to c but $(f(x_n))$ does not converge to L.
2. f does not have a limit at c if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that (x_n) converges to c but $(f(x_n))$ does not converge.

Proof

1. This follows directly from the sequential criterion theorem (123). It is the contrapositive of Theorem 123, which we proved using the contrapositive, i.e. we directly proved Corollary 124 in the proof of Theorem 123
2. This follows directly from part 1. If $(f(x_n))$ does not converge to any L then, by 1, f does not converge to any L, i.e. f does not have a limit.

□

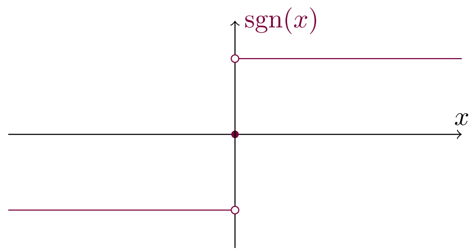
Example 125

1. $\lim_{x\to 0} \frac{1}{x}$ does not exist.
   $(x_n) = (\frac{1}{n})$ converges to 0, but $f(x_n) = \frac{1}{\frac{1}{n}} = n$ does not converge. Therefore Corollary 124.2 shows that this limit does not exist in $\mathbb{R}$. The "in $\mathbb{R}$" emphasizes that we can (and will) extend the concept of limits later to include functions tending to ∞. But even then the limit $\lim_{x\to 0} \frac{1}{x}$ does not exist!

2. The sign/signum function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

does not have a limit at 0.



Let $x_n = \frac{(-1)^n}{n}$, then $\lim(x_n) = 0$, but $\operatorname{sgn}(x_n) = (-1, 1, -1, 1, \dots)$, which does not converge. Therefore Corollary 124.2 shows that this limit does not exist.

4.2 Limit Theorems

Theorem 126 (local boundedness)
Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and $f(x)$ converge for $x \to c$. Then there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that

$$|f(x)| \leq M$$

for all $x \in A \cap V_\delta(c)$.

draw picture

Proof
If $\lim_{x \to c} f(x) = L$, then for $\varepsilon = 1$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < 1$. Therefore

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < 1 + |L|$$

for $x \in A \cap V_\delta(c) \setminus \{c\}$.

- If $c \notin A$, i.e. $f(c)$ is not defined, then setting $M = 1 + |L|$
- If $c \in A$, i.e. $f(c)$ is defined, then setting $M = \max\{|f(c)|, 1 + |L|\}$

yields

$$|f(x)| \leq M$$

for all $x \in A \cap V_\delta(c)$. □

Theorem 127 (Linearity of Limits)
Let $f, g : A \subset \mathbb{R} \to \mathbb{R}$, $\lim_{x\to c} f(x) = F$, $\lim_{x\to c} g(x) = G$ and $b \in \mathbb{R}$. Then

$$\begin{aligned}\lim_{x\to c}(f+g) &= F+G\\ \lim_{x\to c}(f-g) &= F-G\\ \lim_{x\to c}(fg) &= FG\\ \lim_{x\to c}(bf) &= bF\end{aligned}$$

If additionally $g(x) \neq 0$ for all $x \in A$ and $G \neq 0$, then

$$\lim_{x\to c}\left(\frac{f}{g}\right) = \frac{F}{G}$$

Proof
The proof follows directly from the Sequential Criterion Theorem (123) and the linearity of limits theorem for sequences (64). □

It might be a good exercise to prove $\lim_{x\to c}(fg) = FG$ and $\lim_{x\to c}\left(\frac{F}{G}\right)$ from the definition.

Lecture 25 (March 19)

Example 128

- For $c > 0$ Theorem 127 shows

$$\lim_{x\to c}\frac{1}{x} = \frac{1}{\lim_{x\to c} x} = \frac{1}{c}.$$

- $\lim_{x\to 2}(x^2+1)(x^3-4) = 20$ since by Theorem 127

$$\begin{aligned}\lim_{x\to 2}(x^2+1)(x^3-4) &= \lim_{x\to 2}(x^2+1)\lim_{x\to 2}(x^3-4)\\ &= \left((\lim_{x\to 2} x)^2+1\right)\left((\lim_{x\to 2} x)^3-4\right) = (2^2+1)(2^3-4)\\ &= 5\cdot 4 = 20\end{aligned}$$

- $\lim_{x\to 2}\frac{x^2-4}{3x-6} = \frac{4}{3}$
  We can not apply Theorem 127 since

$$G = \lim_{x\to 2} g(x) = \lim_{x\to 2}(3x-6) = 0$$

but for $x \neq 2$,

$$\frac{x^2-4}{3x-6} = \frac{(x+2)(x-2)}{3(x-2)} = \frac{x+2}{3},$$

so for $x \neq 2$, which are the only x relevant in the definition of the limit (118) the functions coincide, i.e. their limit is the same. Therefore

$$\lim_{x\to 2} \frac{x^2-4}{3x-6} = \lim_{x\to 2} \frac{x+2}{3} = \frac{4}{3},$$

even though the original function $\frac{x^2-4}{3x-6}$ is not defined for $x = 2$.

- If $f(x)$ is any polynomial, i.e. $f(x) = \sum_{i=0}^{k} a_i x^i = a_0 + a_1 x + \cdots + a_k x^k$ then

$$\lim_{x\to c} f(x) = f(c),$$

since by linearity

$$\lim_{x\to c} f(x) = \lim_{x\to c} \sum_{i=0}^{k} a_i x^i = \sum_{i=0}^{k} a_i \lim_{x\to c}(x^i) = \sum_{i=0}^{k} a_i (\lim_{x\to c} x)^i = \sum_{i=0} a_i c^i = f(c)$$

- If p, q are polynomials on $\mathbb{R}$ and $q(c) \neq 0$, then

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

Since $q(c) \neq 0$, taking $\varepsilon = \frac{|q(c)|}{2}$, there exists $\delta > 0$ such that for all $|x - c| < \delta$ one has $q(x) \neq 0$. Therefore we can restrict ourselves to those x and then the linearity theorem 127 and the previous example shows that

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

Theorem 129
Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and c be a cluster point of A. If

$$a \leq f(x) \leq b$$

for all $x \in A$ with $x \neq c$ and $\lim_{x\to c} f(x)$ exists, then

$$a \leq \lim_{x\to c} f(x) \leq b.$$

Proof
By the Sequential Criterion Theorem 123 $\lim_{x\to c} f(x) = L$ is equivalent to $(f(x_n))$ converging to L for every sequence (x_n) in A that converges to c with $x_n \neq c$. Since $a \leq f(x) \leq b$ for all $c \neq x \in A$, the sequence $(f(x_n))$ also satisfies

$$a \leq f(x_n) \leq b,$$

which by Theorem 67 , which was the equivalent for sequences, yields

$$a \leq \lim(f(x_n)) \leq b.$$

Therefore $a \leq \lim_{x\to c} f(x) = L = \lim_{n\to\infty}(f(x_n)) \leq b$. □

Theorem 130 (Sandwich/Squeeze Theorem)
Let $f, g, h : A \subset \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in A$ with $x \neq c$ and $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = L$.

Proof
Again by the Sequential Criterion Theorem 123, $\lim_{x\to c} f(x) = L = \lim_{x\to c} g(x)$ is equivalent to every sequence (x_n) in A with $x_n \neq c$ converging to c satisfying

$$\lim_{n\to\infty} f(x_n) = L = \lim_{n\to\infty} h(x_n).$$

By the condition

$$f(x_n) \leq g(x_n) \leq h(x_n)$$

for every such sequence and all $n \in \mathbb{N}$. Therefore the sandwich theorem for sequences yields

$$\lim_{n\to\infty}(g(x_n)) = L$$

for all such sequences (x_n), implying $\lim_{x\to c} g(x) = L$. □

Example 131

1. $\lim_{x\to 0} x^{\frac{3}{2}} = 0$ ($x^{\frac{3}{2}}$ is only defined for $x > 0$)
   For $0 < x \leq 1$ one has $x \leq x^{\frac{1}{2}} \leq 1$ and therefore

   $$x^2 \leq x^{\frac{3}{2}} \leq x.$$

   Previously we had $\lim_{x\to 0} x^2 = 0$, $\lim_{x\to 0} x = 0$, implying

   $$\lim_{x\to 0} x^{\frac{3}{2}} = 0$$

   by the Sandwich Theorem.

2. Another one in Quiz.

Theorem 132
Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. If

$$\lim_{x \to c} f(x) > 0,$$

then there exists a neighborhood $V_\delta(c)$ of c such that $f(x) > 0$ for all $x \in A \cap V_\delta(c) \setminus \{c\}$. Similarly "$< 0$".

This holds without any continuity assumptions on f

Proof
Let $L = \lim_{x \to c} f(x) > 0$. Then for $\varepsilon = \frac{1}{2}L$ there exists a $\delta > 0$ such that for $x \in A$ satisfying $0 < |x - c| < \delta$

$$|f(x) - L| < \frac{1}{2}L \quad \Leftrightarrow \quad -\frac{1}{2}L < f(x) - L < \frac{1}{2}L \quad \Longrightarrow \quad 0 < \frac{1}{2}L < f(x).$$

□

4.3 Extensions of the Limit Concept

The previous definitions can now be easily extended to other concepts of limits.

Definition 133 (One-Sided Limits)
Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of

- $\{x \in A \,|\, x > c\}$. Then $\lim_{x \to c^+} f(x) = L$ if $L \in \mathbb{R}$ exists such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < x - c < \delta$ it holds $|f(x) - L| < \varepsilon$.
- $\{x \in A \,|\, x < c\}$. Then $\lim_{x \to c^-} f(x) = L$ if $L \in \mathbb{R}$ exists such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$ it holds $|f(x) - L| < \varepsilon$.

Definition 134 (Infinite Limits)
Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Then

- $\lim_{x \to c} f(x) = \infty$ if for every $r \in \mathbb{R}$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$ it holds $f(x) > r$.
- $\lim_{x \to c} f(x) = -\infty$ if for every $r \in \mathbb{R}$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$ it holds $f(x) < r$.

Definition 135 (Limits at Infinity)
Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and suppose

- $(a, \infty) \subset A$ for some $a \in \mathbb{R}$. Then $\lim_{x \to \infty} f(x) = L$ if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $K > a$ such that for all $x > K$ it holds $|f(x) - L| < \varepsilon$.

- $(-\infty, a) \subset A$ for some $a \in \mathbb{R}$. Then $\lim_{x\to-\infty} f(x) = L$ if there exists $L \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $K < a$ such that for all $x < K$ it holds $|f(x) - L| < \varepsilon$.

Lecture 26 (March 20)

5 Continuity

5.1 Continuous Functions

Definition 136 (Epsilon-Delta Criterion)
Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and $c \in A$. f is continuous at c if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ satisfies $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.
If f is not continuous at c, then it is called discontinuous at c.

This is the same as $\lim_{x\to c} f(x) = f(c)$ without the technical assumptions in order for a limit to make sense. Draw picture!

Corollary 137
If $c \in A$ is a cluster point, then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$

Remarks 138

- To be continuous at c
  - f must be defined at c
  - the limit can not be infinity.
- If c is not a cluster point of A, then f can still be continuous in c. In fact f is automatically continuous in c if c is not a cluster point. We will mostly ignore these cases as nothing happens and they are boring.

The following Corollaries directly follow from the previous considerations for limits and sequences.

Corollary 139
$f : A \subset \mathbb{R} \to \mathbb{R}$ is continuous at $c \in A$ if and only if given any ε-neighborhood $V_\varepsilon(f(c))$ there exists a δ-neighborhood $V_\delta(c)$ such that if $x \in A \cap V_\delta(c)$, then $f(x) \in V_\varepsilon(f(c))$, i.e.

$$f(A \cap V_\delta(c)) \subset V_\varepsilon(f(c)).$$

Corollary 140 (Sequential Criterion for Continuity)
$f : A \subset \mathbb{R} \to \mathbb{R}$ is continuous at $c \in A$ if and only if for every sequence (x_n) in A that converges to c the sequence $(f(x_n))$ converges to $f(c)$.

By taking the contrapositive we immediately get the following.

Corollary 141 (Discontinuity Criterion)
$f : A \subset \mathbb{R} \to \mathbb{R}$ is discontinuous at $c \in A$ if and only if there exists a sequence (x_n) in A that converges to c, but $(f(x_n))$ does not converge to $f(c)$.

Definition 142
Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and $B \subset A$. f is continuous on B if f is continuous at every point in B.

Example 143
Most of the following are direct consequences of the limits calculated in Examples 122.

1. Let $b \in \mathbb{R}$. Then $f(x) = b$ is continuous on $\mathbb{R}$.

   For $c \in \mathbb{R}$ one has $\lim_{x\to c} f(x) = \lim_{x\to c} b \overset{122}{=} b = f(c)$.

2. $f(x) = x$ is continuous on $\mathbb{R}$.

   For $c \in \mathbb{R}$ one has $\lim_{x\to c} f(x) = \lim_{x\to c} x \overset{122}{=} c = f(c)$.

3. $f(x) = x^2$ is continuous on $\mathbb{R}$.

   For $c \in \mathbb{R}$ one has $\lim_{x\to c} f(x) = \lim_{x\to c} x^2 \overset{122}{=} c^2 = f(c)$.

4. $f(x) = \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$.

   For $c > 0$ one has $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{1}{x} \overset{122}{=} \frac{1}{c} = f(c)$
   For $c < 0$ note that by the definition of limits (by swapping $x \to -x$ and $c \to -c$), linearity of limits (Theorem 127) and the positive case

   $$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to -c} f(-x) = \lim_{x\to -c} \frac{1}{-x} = \lim_{x\to -c}\left(-\frac{1}{x}\right) \\ &= -\lim_{x\to -c} \frac{1}{x} = -\frac{1}{-c} = \frac{1}{c} = f(c)\end{aligned}$$

5. $f(x) = \frac{1}{x}$ is discontinuous at $c = 0$

   By assumption $f(c)$ has to be defined. Alternatively by 125 the limit $\lim_{x\to 0} \frac{1}{x}$ does not exist. Alternatively consider the sequence $\frac{(-1)^n}{n} \to 0$, but $(f(\frac{(-1)^n}{n})) = (-1, 2, -3, 4, \dots)$ does not converge and the discontinuity criterion yields that f is discontinuous

6. $f(x) = \operatorname{sgn}(x)$ is discontinuous at 0 (and continuous on $\mathbb{R} \setminus \{0\}$)

   - By 125 the limit $\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist. Alternatively consider the sequence $\frac{(-1)^n}{n} \to 0$, but $(f(\frac{(-1)^n}{n})) = (-1, 1, -1, 1, \dots)$ does not converge and the discontinuity criterion yields that f is discontinuous.

7. $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point in $\mathbb{R}$.

Suppose c is irrational. Then by density of $\mathbb{Q}$ in $\mathbb{R}$ for every $\delta > 0$ there exists a $x \in \mathbb{Q}$ such that

$$0 < |x - c| < \delta$$

but

$$|f(x) - f(c)| = |1 - 0| = 1.$$

Similarly for $c \in \mathbb{R} \setminus \mathbb{Q}$.

5.2 Combinations of Continuous Functions

Corollary 144
Let $f, g : A \subset \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$. Suppose $c \in A$ and f and g are continuous at c.

- Then $f + g$, $f - g$, fg, bf are continuous at c.
- If additionally $g(x) \neq 0$ for all $x \in A$, then $\frac{f}{g}$ is continuous at c.

If they are continuous on $B \subset A$ then these combinations are continuous on B too.

This follows immediately by the linearity of limits of functions (Theorem 127). Therefore one has immediately the following.

Corollary 145

- Polynomials are continuous on $\mathbb{R}$.
- Rational functions are continuous everywhere where the are defined.

Without justification: $\sin(x)$, $\cos(x)$, e^x are continuous on $\mathbb{R}$. $\ln x$ is continuous for $x > 0$. $\sqrt{x}$ and x^r for $r \geq 0$ are continuous for $x \geq 0$. But you can use these in examples.

Theorem 146 (Compositions of continuous functions are continuous)
Let $A, B \subset \mathbb{R}$, $f : A \to B$ and $g : B \to \mathbb{R}$. If f is continuous at $c \in A$ and g is continuous at $f(c)$, then $g \circ f : A \to \mathbb{R}$ is continuous at c. $g \circ f(x) = g(f(x))$

Proof
By definition for every $\varepsilon_1 > 0$ there exists an $\delta_1 > 0$ such that for $x \in A$ satisfying

$$|x - c| < \delta_1, \qquad \text{it holds} \qquad |f(x) - f(c)| < \varepsilon_1 \tag{38}$$

and for every $\varepsilon_2 > 0$ there exists an $\delta_2 > 0$ such that for $y \in B$ satisfying

$$|y - f(c)| < \delta_2 \qquad \text{it holds} \qquad |g(y) - g(f(c))| < \varepsilon_2. \tag{39}$$

Given $\varepsilon > 0$, setting $\varepsilon_2 = \varepsilon$ there exists $\delta_2(\varepsilon_2) = \delta_2(\varepsilon)$ and setting $\varepsilon_1 = \delta_2(\varepsilon)$ there exists $\delta = \delta_1$ such that if

$$|x - c| < \delta = \delta_1$$

by (38) it holds

$$|f(x) - f(c)| < \varepsilon_1 = \delta_2$$

and by (39) it holds

$$|g(f(x)) - g(f(c))| < \varepsilon_2 = \varepsilon$$

□

Draw picture!

Lecture 27 (March 24)

5.3 Continuous Functions on Intervals

Definition 147
$f : A \to \mathbb{R}$ is said to be bounded on A if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.
It is unbounded if it is not bounded, i.e. for every $M > 0$ there exists a $x \in A$ such that $|f(x)| > M$.

Theorem 148 (Boundedness Theorem)
Let $I = [a, b]$ be a closed and bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.

Proof
By contradiction.
Suppose f is not bounded. Then for any $n \in \mathbb{N}$ there exists $x_n \in I$ such that

$$|f(x_n)| > n \tag{40}$$

for all $n \in \mathbb{N}$. Since I is bounded the sequence (x_n) is bounded and by Bolzano-Weierstrass there exists a convergent subsequence (x_{n_k}) that converges to some $x \in I$ by Theorem 67 ($a \leq x_n \leq b \implies a \leq \lim(x_n) \leq b$). Since f is continuous $(f(x_{n_k}))$ has to converge by the sequential criterion for continuity. This implies that $(f(x_{n_k}))$ is a bounded sequence (by Theorem 63), which is a contradiction to

$$|f(x_{n_k})| \overset{(40)}{>} n_k \geq k.$$

for all $k \in \mathbb{N}$. $\square$

Remark 149
The conditions in Theorem 148 are crucial as the following counterexamples show.

- Closed Interval: For $I = (0, 1]$ and the continuous on I function $\frac{1}{x}$ is unbounded on I.
- Bounded interval: For $I = \mathbb{R}$ the continuous $f(x) = x$ is unbounded.
- Continuous function: For $I = [-1, 1]$ the function $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is unbounded.

Definition 150 (Absolute Extrema of Functions)
$f : A \subset \mathbb{R} \to \mathbb{R}$ has an

- absolute maximum on A if there exists $x^\star \in A$, called absolute maximum point, such that
$$f(x^\star) \geq f(x)$$
for all $x \in A$.
- absolute minimum on A if there exists $x^\star \in A$, called absolute minimum point, such that
$$f(x^\star) \leq f(x)$$
for all $x \in A$.

Theorem 151 (Extreme Value Theorem)
Let $I = [a, b]$ be a closed and bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

Proof
Consider the nonempty set $f(I) = \{f(x) \,|\, x \in I\}$. Then $f(I)$ is bounded by Theorem 148. As a nonempty, bounded set it has a supremum s by the completeness property.

Since $s = \sup f(I)$, by Lemma 28, for every $n \in \mathbb{N}$ there exists x_n such that

$$s - \frac{1}{n} < f(x_n) \leq s. \tag{41}$$

Since I is bounded, the sequence (x_n) is bounded. Therefore, by Bolzano-Weierstrass, there exists a convergent subsequence (x_{n_k}), converging to some $x^\star \in I$ ($\in I$ by Theorem 67). Since f is continuous on I, it follows that $\lim_{k\to\infty} f(x_{n_k}) = f(x^\star)$. By the Sandwich Theorem (with $(s - \frac{1}{n_k})$, $(f(x_{n_k}))$, (s)) the condition (41) yields

$$s = \lim_{k\to\infty} \left(s - \frac{1}{n_k}\right) \leq f(x^\star) = \lim_{k\to\infty} f(x_{n_k}) \leq \lim_{k\to\infty} (s) = s.$$

Therefore

$$f(x^\star) = \lim_{k\to\infty} (f(x_{n_k})) = s = \sup f(I).$$

For the minimum use $-f$. □

Theorem 152 (Location of Roots Theorem)
Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.

Proof
Without loss of generality assume $f(a) < 0 < f(b)$ (otherwise use $-f$). We will generate a sequence of intervals $I_n = [a_n, b_n]$ by successive bisections. Let $a_1 = a$ and $b_1 = b$ and $p_1 = \frac{a_1+b_1}{2}$, where

- If $f(p_n) = 0$, then take $c = p_n$ and we are done.
- If $f(p_n) > 0$, then set $a_{n+1} = a_n$, $b_{n+1} = p_n$ and $p_{n+1} = \frac{a_{n+1}+b_{n+1}}{2}$.
- If $f(p_n) < 0$, then set $a_{n+1} = p_n$, $b_{n+1} = b_n$ and $p_{n+1} = \frac{a_{n+1}+b_{n+1}}{2}$.

Either this terminates or we find a nested sequence of closed intervals I_n such that

$$f(a_n) < 0, \qquad f(b_n) > 0$$

for all $n \in \mathbb{N}$. By the Nested Intervals Property (Theorem 47) there exists a point c such that $c \in I_n$ for all $n \in \mathbb{N}$.
By construction, the interval length halves every iteration one has, i.e.

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}$$

and therefore

$$b_n - a_n = \frac{b - a}{2^{n-1}},$$

which converges to 0, showing that $\lim a_n = \lim b_n$, by linearity of limits (a_n and b_n are monotone and bounded sequences and therefore convergent). As

$$a_n \leq c \leq b_n$$

and by the Sandwich Theorem

$$\lim a_n \leq \lim c = c \leq \lim b_n \leq \lim a_n,$$

As $f(a_n) < 0$ and $f(b_n) > 0$ for all $n \in \mathbb{N}$ one has by Theorem 67 (since $f(a_n) \leq 0$ and $f(b_n) \geq 0$)

$$0 \leq \lim(f(b_n)) = f(c) = \lim(f(a_n)) \leq 0,$$

i.e. $f(c) = 0$. □

Theorem 153 (Bolzano's Intermediate Value Theorem)
Let I be an interval and $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and $k \in \mathbb{R}$ satisfy

$$f(a) < k < f(b),$$

then there exists a point $c \in I$ in between a and b such that $f(c) = k$.

This shows that we have to pass every value in between $f(a)$ and $f(b)$

Proof
Suppose that $a < b$ and let $g(x) = f(x) - k$. Then $g(a) < 0 < g(b)$ and by the Location of Roots Theorem 152, there exists a point c such that $a < c < b$ and $0 = g(c) = f(c) - k$, i.e. $f(c) = k$. □

The following summarizes the last results.

Theorem 154
Let I be a closed and bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) = \{f(x) \,|\, x \in I\}$ is a closed and bounded interval.

Proof
By the Extreme Value Theorem 151 there exist $x_\star, x^\star \in I$ such that $m := f(x_\star) = \min_{x \in I} f(x)$ and $M := f(x^\star) = \max_{x \in I} f(x)$, i.e. $m, M \in f(I)$. This immediately implies $f(I) \subset [m, M]$.
To show that $(m, M) \subset f(I)$, take $k \in (m, M)$. Then Bolzano's Intermediate Value Theorem 153 with $a = x_\star$ and $b = x^\star$ shows that there exists $c \in [a, b]$ such that $k = f(c) \in f(I)$.
Therefore $f(I) \subset [m, M] \subset f(I)$, concluding the proof. □

5.4 Uniform Continuity

Previously (Definition 136 and Definition 142) Let $f : A \subset \mathbb{R} \to \mathbb{R}$. f is continuous on A if and only if for every $\varepsilon > 0$ and every $c \in A$ there exists $\delta(c, \varepsilon) > 0$ such that for all $x \in A$ such that

$$|x - c| < \delta(\varepsilon, c)$$

it holds

$$|f(x) - f(c)| < \varepsilon.$$

In particular δ may depend on ε and c. For example $f(x) = \frac{1}{x}$ is continuous in every point $c > 0$ even though the δ shrinks as c gets closer to 0.

Given $\varepsilon > 0$ $\delta = \frac{c}{2}\min\{1, \varepsilon c\}$ (compare to Theorem 64.5)

$$|x - c| < \delta \implies \frac{c}{2} < x$$

$$\implies \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c - x}{xc}\right| < \frac{2}{c^2}|c - x| < \varepsilon \tag{42}$$

so as $c \to 0$ one has $\delta \to 0$.

In contrast for $f(x) = 2x$ this is not the case.

There we have that given $\varepsilon > 0$, if $|x - c| < \delta = \frac{\varepsilon}{2}$, then

$$|2x - 2c| = 2|x - c| < \varepsilon.$$

This is an important feature called uniform continuity.

Definition 155
Let $f : A \subset \mathbb{R} \to \mathbb{R}$. f is uniformly continuous on A if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for all $x, c \in A$ satisfying

$$|x - c| < \delta$$

it holds

$$|f(x) - f(c)| < \varepsilon.$$

Lemma 156 (Nonuniform Continuity Criteria)
Let $f : A \subset \mathbb{R} \to \mathbb{R}$. Then the following are equivalent.

1. f is not uniformly continuous on A.
2. There exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exist points $x_\delta, c_\delta \in A$ such that

$$|x_\delta - c_\delta| < \delta$$

and

$$|f(x_\delta) - f(c_\delta)| \geq \varepsilon$$

3. There exists an $\varepsilon > 0$ and two sequences (x_n) and (c_n) in A such that $\lim(x_n - c_n) = 0$ and $|f(x_n) - f(c_n)| > \varepsilon$ for all $n \in \mathbb{N}$.

Proof
2. is the opposite of the definition of uniform continuity, so 2. $\Leftrightarrow$ 1..
2 $\Rightarrow$ 3: Setting $\delta = \frac{1}{n}$, for every $n \in \mathbb{N}$ there exist points $x_n, c_n \in A$ (after relabeling) such that

$$|x_n - c_n| < \frac{1}{n},$$

implying $(x_n - c_n)$ converges to 0, and $|f(x_n) - f(c_n)| \geq \varepsilon$.
3 $\Rightarrow$ 2: By definition of the limit for every $\delta > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - c_n| < \delta$ for all $n \geq N$. In particular such x_δ and c_δ exist and fulfill $|f(x_\delta) - f(c_\delta)| \geq \varepsilon$ by assumption. $\square$

Lecture 28 (March 31)

Example 157
Now we can prove the earlier claim that $f(x) = \frac{1}{x}$ is not uniformly continuous on $x > 0$.

For $x_n = \frac{1}{n}$ and $c_n = \frac{1}{n+1}$ one has

$$\lim(x_n - c_n) = \lim\left(\frac{1}{n} + \frac{1}{n+1}\right) = 0$$

and

$$|f(x_n) - f(c_n)| = \left|\frac{1}{\frac{1}{n}} - \frac{1}{\frac{1}{n+1}}\right| = |n - (n+1)| = 1$$

Theorem 158 (Uniform Continuity Theorem)
Let I be a closed and bounded interval and let $f : I \to \mathbb{R}$ be continuous. Then f is uniformly continuous on I.

Proof
By contradiction.
Assume that f is not uniformly continuous on I. Then by the nonuniform continuity criteria (Lemma 156) there exists $\varepsilon > 0$, (x_n) and (c_n) in I such that $|x_n - c_n| < \frac{1}{n}$ and $|f(x_n) - f(c_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Since I is bounded (x_n) is bounded and therefore by Bolzano-Weierstrass there exists a subsequence (x_{n_k}), converging to some x. Since I is closed by Theorem 67 ($a \leq x_n \leq b \implies a \leq \lim x_n \leq b$) $x \in I$. Similarly, there exists (c_{n_k}) which also converges to x, to x since

$$|c_{n_k} - x| = |c_{n_k} - x_{n_k} + x_{n_k} - x| \leq \underbrace{|c_{n_k} - x_{n_k}|}_{\to 0} + \underbrace{|x_{n_k} - x|}_{\to 0} \to 0.$$

Since f is continuous

$$f(x_{n_k}) - f(c_{n_k}) \to f(x) - f(x) = 0 \tag{43}$$

by the sequential criterion for continuity 140. (43) is a contradiction to $|f(x_n) - f(c_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. □

draw/explain example $f \in L^1$ and $f > 0$ uniformly continuous, then $f \to 0$.

Lipschitz Functions

Definition 159
Let $f : A \subset \mathbb{R} \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(c)| \leq K|x - c| \tag{44}$$

for all $x, c \in A$, then f is a Lipschitz Function.

Remark 160

- Rewriting (44) we get

  $$\left| \frac{f(x) - f(c)}{x - c} \right| \leq K$$

  for all $x, c \in A$. The left side describes the slope of a secant through the graph of the function. (Draw picture!) So a function is Lipschitz on A if its slope is bounded on A.

- Outlook: In functional analysis bounded linear operators are important. A linear operator $T : U \to V$ is bounded if there exists $M > 0$ such that

  $$\|Tz\|_V \leq M\|z\|_U,$$

  for all $z \in U$, which is the same as Definition 159 since T is linear, i.e. $T(x) - T(c) = T(x - c) = T(z)$. The operator norm, "the smallest Lipschitz constant of the operator describes the maximal growth of a operator".

Theorem 161
If $f : A \subset \mathbb{R} \to \mathbb{R}$ is Lipschitz, then f is uniformly continuous on A.

Proof
If there exists $K > 0$ such that

$$|f(x) - f(c)| < K|x - c|,$$

then given $\varepsilon > 0$ taking $\delta = \frac{\varepsilon}{K}$ if $|x - c| < \delta$ one has

$$|f(x) - f(c)| \leq K|x - c| < K\delta = K\frac{\varepsilon}{K} = \varepsilon.$$

□

Example 162

1. Let $b > 0$, then $f : [0, b] \to \mathbb{R}$, $f(x) = x^2$ is Lipschitz.

   One has

   $|f(x) - f(c)| = |x^2 - c^2| = |(x + c)(x - c)| \leq |x + c|\,|x - c| \leq 2b|x - c|,$

   and therefore f has the Lipschitz constant $2b$.

2. Let $g : [0, 2] \to \mathbb{R}, g(x) = \sqrt{x}$ is uniformly continuous, but not Lipschitz.

   (Lip) For $c = 0$ and $x \neq 0$ one has

   $$\left|\frac{g(x) - g(c)}{x - c}\right| = \left|\frac{\sqrt{x} - \sqrt{0}}{x - 0}\right| = \frac{1}{\sqrt{x}} \to \infty$$

   as $x \to 0$.

   (Uni) g is continuous and $[0, 2]$ a closed and bounded interval. Therefore by the Uniform Continuity Theorem 158 is is uniformly continuous.

3. $g : (0, \infty) \to \mathbb{R}$, $g(x) = \sqrt{x}$ is uniformly continuous.

   By the previous example it is uniformly continuous on $[0, 2]$.
   On $[1, \infty)$ one has

   $$|g(x) - g(c)| = |\sqrt{x} - \sqrt{c}| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| \leq \frac{1}{2}|x - c|,$$

   i.e. it is Lipschitz continuous on $[1, \infty)$ and therefore uniformly continuous on $[1, \infty)$.
   Given $\varepsilon > 0$, setting $\delta = \min\{1, \delta_{[0,2]}, \delta_{[1,\infty)}\}$for all $x, c \geq 0$ satisfying $|x - c| < \delta$ one has that $|f(x) - f(c)| < \varepsilon$ if $x, c \in [0, 2]$ or $x, c \in [1, \infty)$ by the two cases. The case $x \in [0, 1)$ and $c \in (2, \infty)$ (or the other way around) is excluded since

   $$|x - c| \leq \delta \leq 1.$$

Lecture 29 (April 02)

Continuous Extension Theorem
Previously

- Continuity on closed, bounded intervals implies uniform continuity (Uniform Continuity Theorem 158)
- $f(x) = \frac{1}{x}$ is (pointwise) continuous on $(0, 1)$, see (42), but not uniformly continuous on $(0, 1)$, see Example 157

When is a function uniformly continuous on an open interval? Before we need the following.

Lemma 163
If $f : A \subset \mathbb{R} \to \mathbb{R}$ is uniformly continuous on A and (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Proof
Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that if $x, y \in A$ satisfy

$$|f(x) - f(y)| < \varepsilon \tag{45}$$

(which is possible since f is uniformly continuous). Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ it holds $|x_n - x_m| < \delta$. Therefore (45)

$$|f(x_n) - f(x_m)| < \varepsilon$$

for $m, n \geq N$, i.e. $f(x_n)$ is Cauchy. □

Theorem 164 (Continuous Extension Theorem)
Let $a, b \in \mathbb{R}$. A function f is uniformly continuous on (a, b) if and only if it can be defined at a and b such that the extended function is continuous on $[a, b]$.

Proof
" $\Longleftarrow$ ": The extended function is uniformly continuous by the Uniform Continuity Theorem 158. Therefore f is uniformly continuous.
" $\Longrightarrow$ ": Suppose f is uniformly continuous on (a, b). We will show that $\lim_{x \to a} f(x)$ (as a one-sided limit) exists. Let (x_n) be a sequence in (a, b) such that $\lim(x_n) = a$. Since real sequences are convergent if and only if they are Cauchy (Theorem 96), (x_n) is Cauchy and therefore by the previous Lemma 163 $f(x_n)$ is Cauchy sequence and therefore converging to some $L \in \mathbb{R}$, i.e. $\lim f(x_n) = L$. Let y_n be any other sequence in (a, b) such that $y_n \to a$. Then $\lim(x_n - y_n) = a - a = 0$ and therefore by uniform continuity $\lim(f(x_n) - f(y_n)) = 0$, implying

$$f(y_n) = \underbrace{f(y_n) - f(x_n)}_{\to 0} + \underbrace{f(x_n)}_{\to L} \to L$$

for $n \to \infty$, i.e. $\lim(f(y_n)) = L$. So every sequences (x_n) in A that converges to a satisfy $f(x_n)$ converges to L, which by the Sequential Criterion Theorem 123 yields that $\lim_{x \to a} f(x) = L$. Therefore extending

$$\tilde{f}(x) = \begin{cases} L & \text{for } x = a \\ f(x) & \text{for } x \in (a, b) \end{cases}$$

shows $\tilde{f}$ is continuous on $[a, b)$. Analogously one can show that $\lim_{x \to b} f(x)$ exists, proving the claim. □

End of material for the final Exam.
Lecture 30 (April 03)

6 Outlook

6.1 Differentation

Definition 165
Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $c \in I$. $f'(c)$ is the derivative of f at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

exists. In that case f is differentiable at c. If f is differentiable for all $c \in I$, then $f' : I \to \mathbb{R}$, $f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$ is the derivative of f.

Theorem 166 (Differentiability implies Continuity)
If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous in c.

Sketch of proof For $x \neq c$

$$f(x) - f(c) = \underbrace{\frac{f(x) - f(c)}{x - c}}_{\to f'(c)} \underbrace{(x - c)}_{\to 0} \to 0$$

□

Example 167

- $f(x) = x^2$ is everywhere differentiable and $f'(x) = 2x$

  $$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x} \frac{y^2 - x^2}{y - x} = \lim_{y \to x} \frac{(y - x)(y + x)}{y - x} = \lim_{y \to x} (y + x) = 2x$$

  Similarly using the Binomial Theorem (Proposition 81) $(x^n)' = nx^{n-1}$ for $n \in \mathbb{N}$ and setting $h = y - x$

- Chain rule

  $h(x) = f(g(x))$, where f, g are differentiable, then

  $$\begin{aligned} h'(x) &= \lim_{y \to x} \frac{h(y) - h(x)}{y - x} = \lim_{y \to x} \left(\frac{f(g(y)) - f((g(x)))}{y - x} \right) \\ &= \lim_{y \to x} \Bigg(\underbrace{\frac{f(g(y)) - f((g(x)))}{g(y) - g(x)}}_{\to f'(g(x))} \underbrace{\frac{g(y) - g(x)}{y - x}}_{g'(x)} \Bigg) = f'(g(x))h'(x) \end{aligned}$$

- $f(x) = |x|$ is continuous but not differentiable at $x = 0$

  $$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1 \neq -1 = \lim_{x \to 0^-} \frac{-x}{x} \\ &= \lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} \end{aligned}$$

Theorem 168
If f has a relative extremum at c (inside an open interval) and $f'(c)$ exists, then $f'(c) = 0$.

Sketch of Proof Assume it is a maximum. If $f'(c) \gtrless 0$, then in a neighborhood of c

$$f(x) - f(c) = (x - c) \underbrace{\frac{f(x) - f(c)}{x - c}}_{\gtrless 0} > 0$$

for $x \gtrless c$, i.e. $f(x) > f(c)$, so f does not have a maximum, a contradiction. □

This also shows that increasing ⇔ positive derivative.

Theorem 169 (Mean Value Theorem)
Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Sketch of Proof

1. Special case $f(a) = f(b)$: If f is constant then every point fulfills it, otherwise it attains a maximum/minimum by the extreme value theorem at some point where $f' = 0$ by the previous theorem

2. For $f(a) \neq f(b)$, the function

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \tag{46}$$

satisfies $\varphi(a) = 0 = \varphi(b)$, therefore by the first case, there exists c such that $0 = \varphi'(c) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

□

6.2 Approximation

The derivative approximates the function, via

$$f(x) \approx f(a) + f'(a)(x - a)$$

if f' is continuous. It uses the functions slope at a point. Draw picture! When using higher derivatives one can approximate f better.

Theorem 170 (Taylor Approximation)
Let $n \in \mathbb{N}$, $f, f', f'', \ldots, f^{(n)}$ be continuous on $[a, b] \subset \mathbb{R}$, $f^{(n+1)}$ exist on (a, b) and $x_0 \in [a, b]$. Then for any $x \in [a, b]$, there exists c in between x and x_0 such that

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}_{n\text{-th Taylor polynom}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}}_{\text{remainder}}$$

The proof is a higher order (meaning more derivatives) version of what was done in (46).

6.3 Sequences of Functions

Define for every $n \in \mathbb{N}$ the function $f_n : A \to \mathbb{R}$. Then (f_n) is a sequence of functions and it converges pointwise on A if for every $x \in A$ it holds $(f_n(x))$ converges to some $f(x) \in \mathbb{R}$. So it converges in every point.
This is equivalent to

Definition 171 (Pointwise Convergence)
$f_n : A \to \mathbb{R}$ converges pointwise to $f : A \to \mathbb{R}$ if for every ε and $x \in A$ there exists a $N(\varepsilon, x) \in \mathbb{N}$ such that if $n \geq N$ then

$$|f_n(x) - f(x)| < \varepsilon.$$

Similar to uniform continuity

Definition 172 (Uniform Convergence)
$f_n : A \to \mathbb{R}$ converges uniformly to $f : A \to \mathbb{R}$ if for every ε there exists a $N(\varepsilon) \in \mathbb{N}$ such that if $n \geq N$ then

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in A$.

In pointwise convergence continuity is not preserved, but if the convergence is uniform then continuity is preserved.

Example 173
draw picture!

$$f_n(x) = \begin{cases} -1 & \text{for } x < -\frac{1}{n} \\ nx & \text{for } -\frac{1}{n} < x < \frac{1}{n} \\ 1 & \text{for } \frac{1}{n} < x \end{cases}$$

is continuous but not uniformly continuous and

$$f_n(x) \to \operatorname{sgn}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } 0 < x \end{cases}$$

is not continuous.

There are similar results for the preservation of differentiability and integrability.

6.4 Integration

A tagged partition $\mathcal{P}$ is a splitting points x_i together with sampling points t_i of an interval (a, b) such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \qquad \text{and} \qquad t_i \in [x_{i-1}, x_i].$$

It is δ-fine if the maximal distance of this splitting is finer then δ, i.e.

$$\max_{1 \leq i \leq n} (x_i - x_{i-1}) < \delta$$

Definition 174
R is called the Riemann integral of f over $[a, b]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any tagged partition $\mathcal{P}$ that is δ-fine and $t_i \in [x_{i-1}, x_i]$ it holds

$$\Bigg| \underbrace{\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})}_{\text{Riemann Sum}} - R \Bigg| < \varepsilon,$$

i.e. the Riemann Sum converges if for $\delta \to 0$ and write $\int_a^b f(x)\ dx = R$.

Draw picture!
This represents the area under the function.

Theorem 175 (Fundamental Theorem Of Calculus)
Suppose there is a finite set E in $[a, b]$ and the functions $f, F : [a, b] \to \mathbb{R}$ satisfy

- F is continuous on $[a, b]$
- $F'(x) = f(x)$ for all $x \in [a, b] \setminus E$
- f is Riemann integrable on $[a, b]$

Then

$$F(b) - F(a) = \int_a^b f(x)\ dx.$$

Rough Sketch of Proof Wlog $E = \varnothing$ (otherwise split $[a, b]$ at every element of E and treat the pieces individually) Then for small δ since by the mean value theorem $\frac{F(x_i)-F(x_{i-1})}{x_i - x_{i-1}} = F'(t_i) = f(t_i)$ for some $t_i \in [x_{i-1}, x_i]$, implying

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - F(b) + F(a) \right| = \left| \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) - F(b) + F(a) \right|$$

$$= |F(b) - F(a) - F(b) + F(a)| = 0$$

Since f is Riemann integrable on $[a,b]$ there exists $\int_a^b f(x)\ dx$ such that

$$\begin{aligned} 0 &= \lim_{\delta\to 0}\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \int_a^b f(x)\ dx\right| \\ &= \lim_{\delta\to 0}\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - F(b) + F(a) + F(b) - F(a) - \int_a^b f(x)\ dx\right| \\ &= F(b) - F(a) - \int_a^b f(x)\ dx. \end{aligned}$$

□

Problem: $\int_a^b f(x)\ dx$ is not defined for unbounded f.
For example for $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$ and $f(0) = 0$ is unbounded, but integrable on $[a,1]$ for every $a > 0$. Then one defines $\int_0^1 \frac{1}{\sqrt{x}}\ dx = \lim_{a\to 0}\int_a^1 \frac{1}{\sqrt{x}}\ dx$ if it exists. Here $\int_a^1 \frac{1}{\sqrt{x}}\ dx = 2x^{\frac{1}{2}}|_a^1 = 2 - 2\sqrt{a}$ and $\int_0^1 \frac{1}{\sqrt{x}}\ dx = \lim_{a\to 0}(2 - 2\sqrt{a}) = 2$
Similar one defines $\int_0^\infty f(x)\ dx$ as the limit if it exists.
More general and what is usually in mathematics referred to as the integral is the Lebesgue Integral, draw picture! which relies on a bit of (Lebesgue) measure, that gives sets a measure (generalizations of length, area volume).

6.5 Further Concepts

- One can generalize this to more dimensions by basically using a norm instead of the absolute value, yielding functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and having to care for directional problems as seen in Quiz 5.
- One can then extend these concepts to infinite dimensional spaces in functional analysis
- Or one can extend it to curved spaces in differential geometry

But in the end these are all just fancy extensions of the limit concept introduced here.

References

Bartle, Robert G. and Donald R. Sherbert (2011). *Introduction to Real Analysis.* John Wiley & Sons.